11.1 The Square Root Property and Completing the Square

Review of Quadratic Equations and Functions

Following is a summary of what you have already studied about quadratic equations and quadratic functions.

1. A quadratic equation in x can be written in the standard form

$$ax^2 + bx + c = 0, \quad a \neq 0$$

2. Some quadratic equations can be solved by factoring.
3. The polynomial function of the form

$$f(x) = ax^2 + bx + c, \ a \neq 0$$

is a quadratic function. Graphs of quadratic functions are called parabolas. The shape of the graph is cuplike.

4. The real solutions of $ax^2 + bx + c = 0$ correspond to the x-intercepts for the graph of the quadratic function $f(x) = ax^2 + bx + c$.

Example 1: Consider the quadratic function given by

$$f(x) = 2x^2 - 9x + 4.$$

Find the x-intercepts of the graph. $(\frac{1}{2}, 0)$ & $(4, 0)$

$f(x)=0$

Solve: $0 = 2x^2 - 9x + 4$

$0 = (2x - 1)(x - 4)$

Either $2x - 1 = 0$, or $x - 4 = 0$

SDWK: 4; 1,4; 2,2

$1 + 2x - 1 = 1 + 0$

$2x = 1$

$\frac{2x}{2} = \frac{1}{2}$

$x = \frac{1}{2}$

$4 + x - 4 = 4 + 0$

$x = 4$

The Square Root Property

If u is an algebraic expression and d is a nonzero real number, then $u^2 = d$ has exactly two solutions:

$$\text{If } u^2 = d, \text{ then } u = \sqrt{d} \text{ or } u = -\sqrt{d}.$$

Equivalently,

$$\text{If } u^2 = d, \text{ then } u = \pm\sqrt{d}.$$

This property can be used to solve quadratic equations that are written in the form $u^2 = d$.

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Example 2: Solve the following quadratic equations by using the square root property.

a. $5x^2 = 125$

$x^2 = 25$ Divide both sides by 5 to isolate x^2.

$x = \pm\sqrt{25}$ Apply the square root property.

$x = \pm 5$ Simplify.

Answer: $\{-5, 5\}$

b. $7x^2 = 64$

$\frac{7x^2}{7} = \frac{64}{7}$

$x^2 = \frac{64}{7}$

Either $x = +\sqrt{\frac{64}{7}}$, or $x = -\sqrt{\frac{64}{7}}$

$x = \frac{\sqrt{64}}{\sqrt{7}}$, or $x = \frac{-\sqrt{64}}{\sqrt{7}}$

$x = \frac{8}{\sqrt{7}}$, or $x = \frac{-8}{\sqrt{7}}$

$x = \frac{8}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$, or $x = \frac{-8}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$

$x = \frac{8\sqrt{7}}{7}$, or $x = \frac{-8\sqrt{7}}{7}$

check:

$7\left(\frac{8\sqrt{7}}{7}\right)^2 \stackrel{?}{=} 64$	$7\left(-\frac{8\sqrt{7}}{7}\right)^2 = 64$
$\frac{7 \cdot 8^2(\sqrt{7})^2}{7^2} = 64$	$\frac{7(-8)^2(\sqrt{7})^2}{7^2} = 64$
$\frac{7 \cdot 64 \cdot 7}{7 \cdot 7} = 64$	$\frac{7 \cdot 64 \cdot 7}{7 \cdot 7} = 64$
$64 = 64$	$64 = 64$
TRUE!	TRUE!

The solution set is $\left\{\frac{8\sqrt{7}}{7}, \frac{-8\sqrt{7}}{7}\right\}$

c. $2x^2 - 11 = 0$

$11 + 2x^2 - 11 = 11 + 0$

$2x^2 = 11$

$\frac{2x^2}{2} = \frac{11}{2}$

$x^2 = \frac{11}{2}$

Either $x = \sqrt{\frac{11}{2}}$, or $x = -\sqrt{\frac{11}{2}}$

$x = \frac{\sqrt{11}}{\sqrt{2}}$, or $x = \frac{-\sqrt{11}}{\sqrt{2}}$

$x = \frac{\sqrt{11}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$, or $x = \frac{-\sqrt{11}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$x = \frac{\sqrt{22}}{2}$, or $x = \frac{-\sqrt{22}}{2}$

$\boxed{\text{The solution set is } \left\{\frac{\sqrt{22}}{2}, \frac{-\sqrt{22}}{2}\right\}.}$

check:

$2\left(\frac{\sqrt{22}}{2}\right)^2 - 11 = 0$	$2\left(\frac{-\sqrt{22}}{2}\right)^2 - 11 = 0$
$\frac{2 \cdot (\sqrt{22})^2}{2^2} - 11 = 0$	$\frac{2 \cdot (-\sqrt{22})^2}{2^2} - 11 = 0$
$\frac{2 \cdot 22}{2 \cdot 2} - 11 = 0$	$\frac{2 \cdot 22}{2 \cdot 2} - 11 = 0$
$\frac{2 \cdot 2 \cdot 11}{2 \cdot 2 \cdot 1} - 11 = 0$	$\frac{2 \cdot 2 \cdot 11}{2 \cdot 2} - 11 = 0$
$11 - 11 = 0$	$11 - 11 = 0$
$0 = 0$	$0 = 0$
TRUE!	TRUE!

d. $3x^2 + 18 = 0$

$-18 + 3x^2 + 18 = -18 + 0$

$3x^2 = -18$

$\frac{1}{3} \cdot \frac{3x^2}{1} = \frac{1}{3} \cdot \left(\frac{-18}{1}\right)$

$x^2 = -\frac{2 \cdot 3 \cdot 3}{3 \cdot 1}$

$x^2 = -6$

Either $x = +\sqrt{-6}$, or $x = -\sqrt{-6}$

$x = \sqrt{6}\sqrt{-1}$, or $x = -\sqrt{6}\sqrt{-1}$

$x = \sqrt{6}\, i$, or $x = -\sqrt{6}\, i$

check:

$3(\sqrt{6}\, i)^2 + 18 = 0$	$3(-\sqrt{6}\, i)^2 + 18 = 0$
$3 \cdot (\sqrt{6})^2 \cdot i^2 + 18 = 0$	$3(-\sqrt{6})^2 (i)^2 + 18 = 0$
$3 \cdot 6 \cdot (-1) + 18 = 0$	$3 \cdot 6 \cdot (-1) + 18 = 0$
$-18 + 18 = 0$	$-18 + 18 = 0$
$0 = 0$	$0 = 0$
TRUE!	TRUE!

SDWK

$18 = 2 \cdot 9$, $9 = 3 \cdot 3$

$18 = 2 \cdot 3 \cdot 3$

$\boxed{\text{The solution set is } \{\sqrt{6}\, i, -\sqrt{6}\, i\}.}$

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e. $(x-3)^2 = 36$

Either

$x-3=\sqrt{36}$, or $x-3=-\sqrt{36}$

$x-3=6$	$x-3=-6$
$3+x-3=3+6$	$3+x-3=-6+3$
$x=9$	$x=-3$

The solution set is $\{9,-3\}$

check:

$[(9)-3]^2=36$	$[(-3)-3]^2=36$
$(6)^2=36$	$(-6)^2=36$
$36=36$	$36=36$
TRUE!	TRUE!

f. $(2x+7)^2 = 10$

Either

$2x+7=+\sqrt{10}$, or $2x+7=-\sqrt{10}$

$-7+2x+7=-7+\sqrt{10}$	$-7+2x+7=-7+(-\sqrt{10})$
$2x=-7+\sqrt{10}$	$2x=-7-\sqrt{10}$
$\frac{1}{2}\cdot\frac{2x}{1}=\frac{1}{2}\cdot\left[\frac{-7+\sqrt{10}}{1}\right]$	$\frac{1}{2}\cdot\frac{2x}{1}=\frac{1}{2}\cdot\left[\frac{-7-\sqrt{10}}{1}\right]$
$x=\frac{-7+\sqrt{10}}{2}$	$x=\frac{-7-\sqrt{10}}{2}$

check:

$\left[\frac{2}{1}\left(\frac{-7+\sqrt{10}}{2}\right)+7\right]^2=10$	$\left[\frac{2}{1}\cdot\left(\frac{-7-\sqrt{10}}{2}\right)+7\right]^2=10$
$[-7+\sqrt{10}+7]^2=10$	$[-7-\sqrt{10}+7]^2=10$
$(\sqrt{10})^2=10$	$(-\sqrt{10})^2=10$
$10=10$	$10=10$
TRUE!	TRUE!

The solution set is $\left\{\frac{-7+\sqrt{10}}{2}, \frac{-7-\sqrt{10}}{2}\right\}$

g. $(4x-3)^2 = -9$

Either

$4x-3=+\sqrt{-9}$, or $4x-3=-\sqrt{-9}$

$4x-3=3i$	$4x-3=-3i$
$3+4x-3=3+3i$	$3+4x-3=3+(-3i)$
$4x=3+3i$	$4x=3-3i$
$\frac{4x}{4}=\frac{3+3i}{4}$	$\frac{4x}{4}=\frac{3-3i}{4}$
$x=\frac{3+3i}{4}$	$x=\frac{3-3i}{4}$

check:

$\left[\frac{4}{1}\left(\frac{3+3i}{4}\right)-3\right]^2=-9$	$\left[\frac{4}{1}\left(\frac{3-3i}{4}\right)-3\right]^2=-9$
$[3+3i-3]^2=-9$	$[3-3i-3]^2=-9$
$(3i)^2=-9$	$(-3i)^2=-9$
$9i^2=-9$	$9i^2=-9$
$9(-1)=-9$	$9(-1)=-9$
$-9=-9$	$-9=-9$
TRUE!	TRUE!

The solution set is $\left\{\frac{3+3i}{4}, \frac{3-3i}{4}\right\}$

"Conjugate Pair"

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Completing the Square

How do you solve a quadratic if the quadratic can't be factored, is not given in the form $u^2 = d$, and can't be rewritten in the $u^2 = d$ form by transposing terms in the equation? Interestingly enough, all quadratics can be rewritten in the $u^2 = d$ form by using a technique called "completing the square".

> Finding the Term Needed to Complete the Square
>
> If $x^2 + bx$ is a binomial, then by adding $\left(\frac{b}{2}\right)^2$, which is the square of half of the coefficient of x, a perfect square trinomial will result. That is,
>
> $$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

Example 3: Find the term needed to complete the square.

$b=2$ a. $x^2 + 2x$ $\quad \left[\frac{1}{2}(2)\right]^2 = [1]^2 = \underline{1}$

$b=5$ b. $x^2 + 5x$ $\quad \left[\frac{1}{2}(5)\right]^2 = \left[\frac{5}{2}\right]^2 = \underline{\frac{25}{4}}$

$b=-7$ c. $x^2 - 7x$ $\quad \left[\frac{1}{2}(-7)\right]^2 = \left[\frac{-7}{2}\right]^2 = \underline{\frac{49}{4}}$

Example 4: Determine if each of the following is a perfect square trinomial. Factor each perfect square trinomial.

a. $x^2 + 6x + 9 = (x+3)(x+3) = (x+3)^2$; Yes

b. $x^2 + 5x + \frac{25}{4} = \left(x+\frac{5}{2}\right)\left(x+\frac{5}{2}\right) = \left(x+\frac{5}{2}\right)^2$; Yes

c. $x^2 + \frac{1}{2}x + \frac{1}{16} = \left(x+\frac{1}{4}\right)\left(x+\frac{1}{4}\right) = \left(x+\frac{1}{4}\right)^2$; Yes

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Check: $(-7)^2+6(-7)-7=0$
$49-42-7=0$
$0=0$
TRUE!

$(1)^2+6(1)-7=0$
$1+6-7=0$
$0=0$
TRUE!

Solving Quadratic Equations by Completing the Square

To solve a quadratic equation by completing the square:
1. Rewrite the equation in the form $x^2+bx=c$.
2. Add to both sides the term needed to complete the square.
3. Factor the perfect square trinomial, and solve the resulting equation by using the square root property.

Example 5: Solve by completing the square.

a. $x^2+6x-7=0$

$x^2+6x \quad =7$ Add 7 to both sides.

$x^2+6x+9=7+9$ Add $\left(\frac{b}{2}\right)^2$ *to* both sides.

$(x+3)^2=16$ Factor the left side.

Now, use the square root property to complete the solution.

Either
$x+3=-\sqrt{16}$, or $x+3=+\sqrt{16}$

$x+3=-4$	$x+3=4$
$-3+x+3=-3+(-4)$	$-3+x+3=-3+4$
$x=-7$	$x=1$

$\boxed{\text{The solution set is } \{-7, 1\}.}$

"b" $=8$; $\left[\frac{1}{2}(8)\right]^2=(4)^2=\underline{16}$

b. $x^2+8x+5=0$

$-5+x^2+8x+5=-5+6$

$x^2+8x=-5$

$x^2+8x+\underline{16}=-5+\underline{16}$

$(x+4)(x+4)=11$

$(x+4)^2=11$

Either
$x+4=-\sqrt{11}$, or $x+4=+\sqrt{11}$

$-4+x+4=-4+(-\sqrt{11})$	$-4+x+4=-4+\sqrt{11}$
$x=-4-\sqrt{11}$	$x=-4+\sqrt{11}$

check
$(-4-\sqrt{11})^2+8(-4-\sqrt{11})+5=0$
$(-4-\sqrt{11})(-4-\sqrt{11})-32-8\sqrt{11}+5=0$
$16+8\sqrt{11}+11-32-8\sqrt{11}+5=0$
$0=0$ TRUE!

$\boxed{\text{The solution set is } \{-4-\sqrt{11}, -4+\sqrt{11}\}.}$

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$\frac{1}{2}\cdot(2x^2+8x+5)=\frac{1}{2}\cdot 0$ ← First Step!

$x^2+4x+\frac{5}{2}=0$

$-\frac{5}{2}+x^2+4x+\frac{5}{2}=-\frac{5}{2}+0$

c. $2x^2+8x+5=0$ (Hint: You must divide by 2 before you complete the square.)

$x^2+4x=\frac{-5}{2}$ ← $\left[\frac{1}{2}(4)\right]^2=(2)^2=4$

$x^2+4x+4=\frac{-5}{2}+\frac{8}{2}$

$(x+2)(x+2)=\frac{3}{2}$

$(x+2)^2=\frac{6}{4}$

Either

$x+2=-\sqrt{\frac{6}{4}}$, or $x+2=+\sqrt{\frac{6}{4}}$

$x+2=\frac{-\sqrt{6}}{\sqrt{4}}$	$x+2=\frac{\sqrt{6}}{\sqrt{4}}$
$x+2=\frac{-\sqrt{6}}{2}$	$x+2=\frac{\sqrt{6}}{2}$
$-2+x+2=-\frac{4}{2}+\left(-\frac{\sqrt{6}}{2}\right)$	$-2+x+2=-\frac{4}{2}+\frac{\sqrt{6}}{2}$
$x=\frac{-4-\sqrt{6}}{2}$	$x=\frac{-4+\sqrt{6}}{2}$

The solution set is $\left\{\frac{-4-\sqrt{6}}{2}, \frac{-4+\sqrt{6}}{2}\right\}$.

Check

$2\left(\frac{-4-\sqrt{6}}{2}\right)^2+8\left(\frac{-4-\sqrt{6}}{2}\right)+5=0$

$2\left(\frac{-4-\sqrt{6}}{2}\right)\left(\frac{-4-\sqrt{6}}{2}\right)+4(-4-\sqrt{6})+5=0$

$\frac{1}{2}[16+8\sqrt{6}+6]-16-4\sqrt{6}+5=0$

$8+4\sqrt{6}+3-16-4\sqrt{6}+5=0$

$0=0$ TRUE!

$2\left(\frac{-4+\sqrt{6}}{2}\right)^2+8\left(\frac{-4+\sqrt{6}}{2}\right)+5=0$

$\frac{2}{1}\cdot\left(\frac{-4+\sqrt{6}}{2}\right)\left(\frac{-4+\sqrt{6}}{2}\right)+4(-4+\sqrt{6})+5=0$

$\frac{1}{2}\cdot(16-8\sqrt{6}+6)-16+4\sqrt{6}+5=0$

$8-4\sqrt{6}+3-11+4\sqrt{6}=0$

$0=0$

TRUE!

Compound Interest Applied Problems

Suppose that an amount of money, P, is invested at interest rate r, compounded annually. In t years, the amount, A, or balance, in the account is given by the formula

$$A=P(1+r)^t$$

Example 6: You invested \$3000 in an account whose interest is compounded annually. After 2 years, the amount, or balance, in the account is \$4320. Find the annual interest rate.

A = \$4,320
t = 2
P = \$3,000

$4{,}320=3{,}000[1+r]^2$

$\frac{4{,}320}{3{,}000}=\frac{3{,}000(1+r)^2}{3000}$

$\frac{36}{25}=(1+r)^2$

Either

$1+r=-\sqrt{\frac{36}{25}}$, or $1+r=+\sqrt{\frac{36}{25}}$

$1+r=-\frac{6}{5}$	$1+r=\frac{6}{5}$
$-1+1+r=\frac{-5}{5}+\left(-\frac{6}{5}\right)$	$-1+1+r=-\frac{5}{5}+\frac{6}{5}$
$r=\frac{-11}{5}$ Not possible	$r=\frac{1}{5}$

$r=\frac{1}{5}=\frac{1}{5}\cdot\frac{20}{20}$

$r=\frac{20}{100}$

$r=20\%$

The annual interest rate was 20%.

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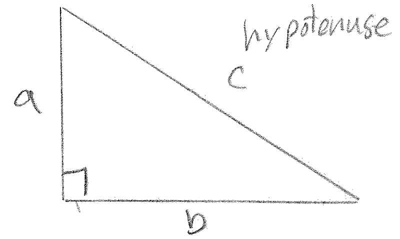


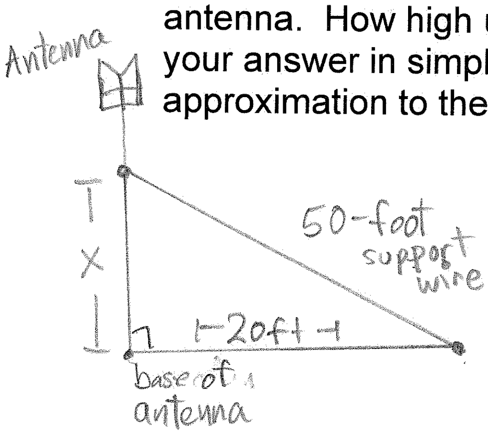
$a^2+b^2=c^2$

Applied Problems Using the Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle equals the square of the length of the hypotenuse. If the legs have length a and b, and the hypotenuse has length c, then

$$a^2+b^2=c^2$$

Example 7: A 50-foot supporting wire is to be attached to an antenna. The wire is anchored 20 feet from the base of the antenna. How high up the antenna is the wire attached? Express your answer in simplified radical form, and then find a decimal approximation to the nearest tenth of a foot.



let x = height along the antenna where the support wire is attached.

$$a^2+b^2=c^2$$

$$(x)^2+(20)^2=(50)^2$$

$$x^2+400=2{,}500$$

$$-400+x^2+400=-400+2{,}500$$

$$x^2=2{,}100$$

Either

$x=-\sqrt{2{,}100}$, or $x=+\sqrt{2{,}100}$

$x=-\sqrt{21}\cdot\sqrt{100}$	$x=\sqrt{21}\cdot\sqrt{100}$
$x=-\sqrt{21}\cdot 10$	$x=\sqrt{21}\cdot 10$
$x=-10\sqrt{21}$	$x=10\sqrt{21}$
Negative Heigth is possible!	$x\approx 10\cdot(4.5826)$
	$x\approx 45.826$
	$x\approx 45.8$ ft

$\boxed{\text{Ans: The support wire is attached 45.8 ft up the antenna.}}$

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Using the Distance Formula

The distance, d, between the points (x_1, y_1) and (x_2, y_2), is given by the Distance Formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 8:

(x_1, y_1) (x_2, y_2)

a. Find the distance between the points (6,−1) and (9,3).

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{[(9) - (6)]^2 + [(3) - (-1)]^2}$

$d = \sqrt{(3)^2 + (4)^2}$

$d = \sqrt{9 + 16}$

$d = \sqrt{25}$

$d = 5$

The distance between the points is 5 units.

b. Find the exact distance between the given points, and then use your calculator to approximate the distance to two decimal places.

(7,4) and (−1,−5)

(x_2, y_2) (x_1, y_1)

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{[(7) - (-1)]^2 + [(4) - (-5)]^2}$

$d = \sqrt{(8)^2 + (9)^2}$

$d = \sqrt{64 + 81}$

$d = \sqrt{145}$

$d \approx 12.0416\ldots$

$d \approx 12.04$

The distance is approximately 12.04 units.

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Answers Section 11.1

Example 1: $(4,0), \left(\frac{1}{2},0\right)$

Example 2:

a. $\{-5,5\}$

b. $\left\{-\frac{8\sqrt{7}}{7}, \frac{8\sqrt{7}}{7}\right\}$

c. $\left\{-\frac{\sqrt{22}}{2}, \frac{\sqrt{22}}{2}\right\}$

d. $\{-i\sqrt{6}, i\sqrt{6}\}$

e. $\{9,-3\}$

f. $\left\{\frac{\sqrt{10}-7}{2}, \frac{-\sqrt{10}-7}{2}\right\}$

g. $\left\{\frac{3+3i}{4}, \frac{3-3i}{4}\right\}$

Example 3:

a. 1

b. $\frac{25}{4}$

c. $\frac{49}{4}$

Example 4:

a. $(x+3)^2$

b. $\left(x+\frac{5}{2}\right)^2$

c. $\left(x+\frac{1}{4}\right)^2$

Example 5:

a. $\{-7,1\}$

b. $\{-4-\sqrt{11}, -4+\sqrt{11}\}$

c. $\left\{\frac{-4-\sqrt{6}}{2}, \frac{-4+\sqrt{6}}{2}\right\}$

Example 6: The annual interest rate is 20%.

Example 7: The wire is attached 45.8 feet up the antenna.

Example 8:

a. 5

b. $\sqrt{145} \approx 12.04$

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11.2 The Quadratic Formula

Solving Quadratic Equations Using the Quadratic Formula.
By solving the general quadratic equation $ax^2 + bx + c = 0$ using the method of completing the square, one can derive the quadratic formula. The quadratic formula can be used to solve any quadratic equation.

> The Quadratic Formula
> The solutions of a quadratic equation in standard form $ax^2 + bx + c = 0$, with $a \neq 0$, are given by the quadratic formula
> $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1: Solve the given quadratic equations by using the quadratic formula.

a. $2x^2 = 6x - 1$

$-6x + 1 + 2x^2 = -6x + 1 + 6x - 1$

$2x^2 - 6x + 1 = 0$

$\boxed{a = 2,\ b = -6,\ c = 1}$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(1)}}{2(2)}$

$x = \dfrac{6 \pm \sqrt{36 - 8}}{4}$

$x = \dfrac{6 \pm \sqrt{28}}{4}$

$x = \dfrac{6 \pm \sqrt{4} \cdot \sqrt{7}}{4}$

$x = \dfrac{6 \pm 2\sqrt{7}}{4}$

$x = \dfrac{\cancel{2} \cdot (3 \pm \sqrt{7})}{\cancel{2} \cdot 2}$

$x = \dfrac{3 \pm \sqrt{7}}{2}$

Either $x = \dfrac{3 + \sqrt{7}}{2}$, or $x = \dfrac{3 - \sqrt{7}}{2}$

Check:

$2 \cdot \left(\dfrac{3 + \sqrt{7}}{2}\right)^2 = 6 \cdot \left(\dfrac{3 + \sqrt{7}}{2}\right) - 1$

$\dfrac{2}{1}\left(\dfrac{3 + \sqrt{7}}{2}\right)\left(\dfrac{3 + \sqrt{7}}{2}\right) = 3 \cdot (3 + \sqrt{7}) - 1$

$\frac{1}{2} \cdot (3 + \sqrt{7})(3 + \sqrt{7}) = 9 + 3\sqrt{7} - 1$

$\frac{1}{2} \cdot [9 + 6\sqrt{7} + 7] = 8 + 3\sqrt{7}$

$\frac{1}{2} \cdot [16 + 6\sqrt{7}] = 8 + 3\sqrt{7}$

$8 + 3\sqrt{7} = 8 + 3\sqrt{7}$ TRUE!

$2 \cdot \left(\dfrac{3 - \sqrt{7}}{2}\right)^2 = 6 \cdot \left(\dfrac{3 - \sqrt{7}}{2}\right) - 1$

$\dfrac{2}{1} \cdot \left(\dfrac{3 - \sqrt{7}}{2}\right)\left(\dfrac{3 - \sqrt{7}}{2}\right) = 3 \cdot (3 - \sqrt{7}) - 1$

$\frac{1}{2} \cdot (3 - \sqrt{7})(3 - \sqrt{7}) = 9 - 3\sqrt{7} - 1$

$\frac{1}{2} \cdot [9 - 6\sqrt{7} + 7] = 8 - 3\sqrt{7}$

$\frac{1}{2} \cdot [16 - 6\sqrt{7}] = 8 - 3\sqrt{7}$

$8 - 3\sqrt{7} = 8 - 3\sqrt{7}$

TRUE!

The solution set is $\left\{ \dfrac{3 + \sqrt{7}}{2}, \dfrac{3 - \sqrt{7}}{2} \right\}$.

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b. $3x^2+5=-6x$

$3x^2+5+6x=-6x+6x$

$3x^2+6x+5=0$

$\boxed{\begin{array}{l}a=3\\ b=6\\ c=5\end{array}}$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-(6)\pm\sqrt{(6)^2-4(3)(5)}}{2(3)}$

$x=\dfrac{-6\pm\sqrt{36-60}}{6}$

$x=\dfrac{-6\pm\sqrt{-24}}{6}$

$x=\dfrac{-6\pm\sqrt{4}\cdot\sqrt{-1}\sqrt{6}}{6}$

$x=\dfrac{-6\pm 2i\sqrt{6}}{6}$

$x=\dfrac{\cancel{2}(-3\pm i\sqrt{6})}{\cancel{2}\cdot 3}$

$x=\dfrac{-3\pm i\sqrt{6}}{3}$

Either

$x=\dfrac{-3+i\sqrt{6}}{3}$, or $x=\dfrac{-3-i\sqrt{6}}{3}$

The solution set is $\left\{\dfrac{-3+i\sqrt{6}}{3},\dfrac{-3-i\sqrt{6}}{3}\right\}$.

check!

$3\left(\dfrac{-3+i\sqrt{6}}{3}\right)^2+5=-6\left(\dfrac{-3+i\sqrt{6}}{3}\right)$

$\cancel{3}\cdot\left(\dfrac{-3+i\sqrt{6}}{\cancel{3}}\right)\left(\dfrac{-3+i\sqrt{6}}{3}\right)+5=-2\cdot(-3+i\sqrt{6})$

$\frac{1}{3}[-3+i\sqrt{6}][-3+i\sqrt{6}]+5=6-2i\sqrt{6}$

$\frac{1}{3}\cdot[9-6i\sqrt{6}+6i^2]+5=6-2i\sqrt{6}$

$\frac{1}{3}\cdot[9-6i\sqrt{6}+6(-1)]+5=6-2i\sqrt{6}$

$\frac{1}{3}\cdot[9-6i\sqrt{6}-6]+5=6-2i\sqrt{6}$

$\frac{1}{3}\cdot[3-6i\sqrt{6}]+5=6-2i\sqrt{6}$

$1-2i\sqrt{6}+5=6-2i\sqrt{6}$

$6-2i\sqrt{6}=6-2i\sqrt{6}$

TRUE!

$3\left(\dfrac{-3-i\sqrt{6}}{3}\right)^2+5=-6\left(\dfrac{-3-i\sqrt{6}}{3}\right)$

$\cancel{3}\left(\dfrac{-3-i\sqrt{6}}{\cancel{3}}\right)\left(\dfrac{-3-i\sqrt{6}}{3}\right)+5=-2[-3-i\sqrt{6}]$

$\frac{1}{3}\cdot(-3-i\sqrt{6})(-3-i\sqrt{6})+5=6+2i\sqrt{6}$

$\frac{1}{3}\cdot(9+6i\sqrt{6}+6i^2)+5=6+2i\sqrt{6}$

$\frac{1}{3}(9+6i\sqrt{6}+6(-1))+5=6+2i\sqrt{6}$

$\frac{1}{3}(9+6i\sqrt{6}-6)+5=6+2i\sqrt{6}$

$\frac{1}{3}\cdot(3+6i\sqrt{6})+5=6+2i\sqrt{6}$

$1+2i\sqrt{6}+5=6+2i\sqrt{6}$

$6+2i\sqrt{6}=6+2i\sqrt{6}$

TRUE!

LCD $=x^2$

c. $3+\dfrac{4}{x}=-\dfrac{2}{x^2}$

$\dfrac{x^2}{1}\cdot\left[\dfrac{3}{1}+\dfrac{4}{x}\right]=\dfrac{x^2}{1}\cdot\left[-\dfrac{2}{x^2}\right]$

$3x^2+4x=-2$

$2+3x^2+4x=-2+2$

$3x^2+4x+2=0$

$\boxed{\begin{array}{l}a=3\\ b=4\\ c=2\end{array}}$

$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\dfrac{-(4)\pm\sqrt{(4)^2-4(3)(2)}}{2(3)}$

$x=\dfrac{-4\pm\sqrt{16-24}}{6}$

$x=\dfrac{-4\pm\sqrt{-8}}{6}$

$x=\dfrac{-4\pm\sqrt{4}\cdot\sqrt{-1}\sqrt{2}}{6}$

$x=\dfrac{-4\pm 2i\sqrt{2}}{6}$

$x=\dfrac{\cancel{2}\cdot(-2\pm i\sqrt{2})}{\cancel{2}\cdot 3}$

$x=\dfrac{-2\pm i\sqrt{2}}{3}$

Either

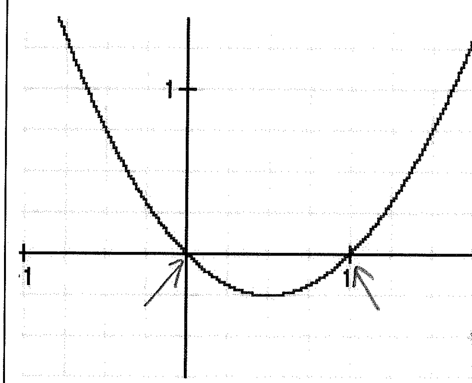
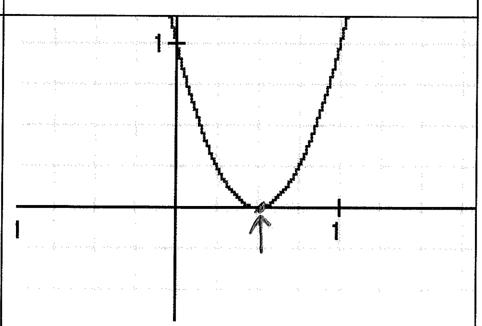
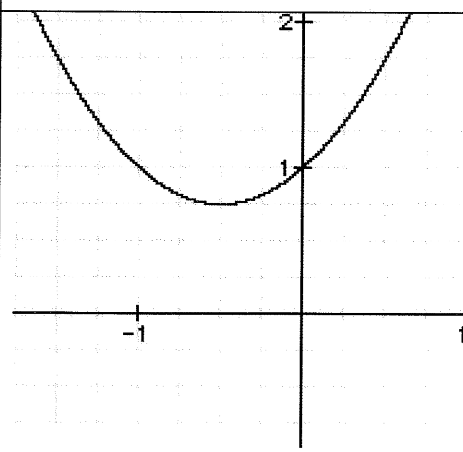
$x=\dfrac{-2-i\sqrt{2}}{3}$, or $x=\dfrac{-2+i\sqrt{2}}{3}$

The solution set is $\left\{\dfrac{-2-i\sqrt{2}}{3},\dfrac{-2+i\sqrt{2}}{3}\right\}$.

The Discriminant

The quantity $b^2 - 4ac$, which appears under the radical sign in the quadratic formula, is called the discriminant. The value of the discriminant for a given quadratic equation can be used to determine the kinds of solutions that the quadratic equation has.

The Discriminant and the Kinds of Solutions to $ax^2 + bx + c = 0$

Value of the Discriminant	**Kinds of Solutions**	**Graph of $y = ax^2 + bx + c$**	
$b^2 - 4ac > 0$	Two unequal real solutions. Graph crosses the x-axis twice.		Each x-intercept corresponds to a solution.
$b^2 - 4ac = 0$	One real solution (a repeated solution) that is a real number. Graph touches the x-axis.		
$b^2 - 4ac < 0$	Two complex solutions that are not real and are complex conjugates of one another. Graph does not touch or cross the x-axis.		No x-intercepts corresponds to "No solutions."

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"discriminant"

Let $D = b^2 - 4ac$

Example 2: For each equation, compute the discriminant. Then determine the number and types of solutions.

a. $x^2 + 6x + 9 = 0$

One Real Solution (Repeated)

$a = 1$
$b = 6$
$c = 9$

$D = b^2 - 4ac$
$D = (6)^2 - 4(1)(9)$
$D = 36 - 36$
$\boxed{D = 0}$

b. $2x^2 - 7x - 4 = 0$

Two unequal Real Solutions

$a = 1$
$b = -7$
$c = -4$

$D = b^2 - 4ac$
$D = (-7)^2 - 4(1)(-4)$
$D = 49 + 16$
$D = 65$
$\boxed{D > 0}$

c. $3x^2 - 2x + 4 = 0$

Two complex Solutions

These solutions are Not Real Numbers.

These two complex Solutions are conjugates of one another.

$a = 3$
$b = -2$
$c = 4$

$D = b^2 - 4ac$
$D = (-2)^2 - 4(3)(4)$
$D = 4 - 48$
$D = -44$
$\boxed{D < 0}$

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Determining Which Method to Use To Solve a Quadratic Equation

Use the following chart as a guide to help you in finding the most efficient method to use to solve a given quadratic equation.

Method 1: $ax^2+bx+c=0$ and ax^2+bx+c can be factored easily	Factor and use the zero-product principle.	Ex: $2x^2-3x+1=0$ $(2x-1)(x-1)=0$ $x=\frac{1}{2},\ x=1$
Method 2: $ax^2+c=0$ The quadratic equation has no x-term.	Solve for x^2 and use the square root property.	Ex: $2x^2-18=0$ $2x^2=18$ $x^2=9$ $x=\pm3$
Method 3: $u^2=d$ and u is a first degree polynomial	Use the square root property	Ex: $(2x-1)^2=9$ $2x-1=\pm3$ $2x=1\pm3$ $x=2,-1$
Method 4: $ax^2+bx+c=0$ and ax^2+bx+c cannot be factored or the factoring is too difficult	Use the quadratic formula.	Ex: $x^2+x+2=0$ $x=\frac{-1\pm\sqrt{(1)^2-4(1)(2)}}{2(1)}$ $x=\frac{-1\pm i\sqrt{7}}{2}$

Example 3: Match each equation with the proper technique given in the chart. Place the equation in the chart and solve it.

a. $(2x-3)^2=7$ ← use "Method 3"

b. $4x^2=-9$ ← use "Method 2"

c. $2x^2+3x=1$ ← use "Method 4"

d. $2x^2+3x=-1$ ← use "Method 1"

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Writing Quadratic Equations from Solutions

To find a quadratic equation that has a given solution set $\{a,b\}$, write the equation $(x-a)(x-b)=0$ and multiply and simplify.

Example 4: Find a quadratic equation that has the given solution set.

a. $\{-2,5\}$

$x=-2$ or $x=5$
$x+2=0$, or $x-5=0$
$(x+2)(x-5)=0$
$x^2-5x+2x-10=0$
$\underline{x^2-3x-10=0}$

b. $\left\{-\frac{1}{2},\frac{2}{5}\right\}$

$x=-\frac{1}{2}$, or $x=\frac{2}{5}$
$2x=-1$, or $5x=2$
$2x+1=0$, or $5x-2=0$
$(2x+1)(5x-2)=0$
$10x^2-4x+5x+2=0$
$\underline{10x^2+x+2=0}$

c. $\{3i,-3i\}$

$x=3i$, or $x=-3i$
$x-3i=0$, or $x+3i=0$
$(x-3i)(x+3i)=0$
$x^2+3ix-3ix-9i^2=0$
$x^2-9(-1)=0$
$\underline{x^2+9=0}$

Applications of Quadratic Equations

Use your calculator to assist you in solving the following problem. Round your answer(s) to the nearest whole number.

Example 5: The number of fatal vehicle crashes per 100 million miles, f(x), for drivers of age x can be modeled by the quadratic function

$$f(x)=0.013x^2-1.19x+28.24$$, $f(x)=3$, solve for x:

What age groups are expected to be involved in 3 fatal crashes per 100 million miles driven?

Solve for x:

$3=0.013x^2-1.19x+28.24$
$-3+3=0.013x^2-1.19x+28.24-3$
$0=0.013x^2-1.19x+25.24$

$\boxed{a=0.013 \quad b=-1.19 \quad c=25.24}$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-(-1.19)\pm\sqrt{(-1.19)^2-4(0.013)(25.24)}}{2(0.013)}$

$x\approx\frac{1.19\pm\sqrt{1.4161-1.31248}}{0.026}$

$x\approx\frac{1.19\pm\sqrt{0.10326}}{0.026}$

$x\approx\frac{1.19\pm0.3219}{0.026}$

Either

$x\approx\frac{1.19+0.3219}{0.026}$, or $x\approx\frac{1.19-0.3219}{0.026}$

$x\approx\frac{1.5119}{0.026}$ | $x\approx\frac{0.8681}{0.026}$

$x\approx58.15$ | $x\approx33.39$

Ans: Drivers aged 33 and 58 are expected to be involved in 3 fatal crashes per 100 million miles driven.

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Example 6: Use your calculator to approximate the solutions of the following quadratic equations to the nearest tenth.

a. $2.1x^2 - 3.8x - 5.2 = 0$

$a = 2.1$
$b = -3.8$
$c = -5.2$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(-3.8) \pm \sqrt{(-3.8)^2 - 4(2.1)(-5.2)}}{2(2.1)}$

$x = \dfrac{3.8 \pm \sqrt{14.44 + 43.68}}{4.2}$

$x = \dfrac{3.8 \pm \sqrt{58.12}}{4.2}$

$x \approx \dfrac{3.8 \pm 7.6237}{4.2}$

Either

$x \approx \dfrac{3.8 + 7.6237}{4.2}$, or $x \approx \dfrac{3.8 - 7.6237}{4.2}$

$x \approx \dfrac{11.4237}{4.2}$	$x \approx \dfrac{-3.8237}{4.2}$
$x \approx 2.72$	$x \approx -0.91$
$x \approx 2.7$	$x \approx -0.9$

The set of approximated solutions is $\{2.7, -0.9\}$

b. $4.5x^2 - 10.2x + 1.3 = 0$

$a = 4.5$
$b = -10.2$
$c = 1.3$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(-10.2) \pm \sqrt{(-10.2)^2 - 4(4.5)(1.3)}}{2(4.5)}$

$x = \dfrac{10.2 \pm \sqrt{104.04 - 23.4}}{9}$

$x = \dfrac{10.2 \pm \sqrt{80.64}}{9}$

$x \approx \dfrac{10.2 \pm 8.98}{9}$

Either

$x \approx \dfrac{10.2 + 8.98}{9}$, or $x \approx \dfrac{10.2 - 8.98}{9}$

$x \approx \dfrac{19.18}{9}$	$x \approx \dfrac{1.22}{9}$
$x \approx 2.13$	$x \approx 0.1356$
$x \approx 2.1$	$x \approx 0.1$

The set of approximated solutions is $\{2.1, 0.1\}$.

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Answers Section 11.2

Example 1:

a. $\left\{\frac{3+\sqrt{7}}{2}, \frac{3-\sqrt{7}}{2}\right\}$

b. $\left\{\frac{-3+i\sqrt{6}}{3}, \frac{-3-i\sqrt{6}}{3}\right\}$

c. $\left\{\frac{-2+i\sqrt{2}}{3}, \frac{-2-i\sqrt{2}}{3}\right\}$

Example 2:

a. *v*alue of discriminant is 0, one real solution.

b. *v*alue of discriminant is 81, two real solutions.

c. *v*alue of discriminant is -44, two complex solutions that are not real and are complex conjugates of each other.

Example 3:

a. Method 3. $\left\{\frac{3+\sqrt{7}}{2}, \frac{3-\sqrt{7}}{2}\right\}$

b. *M*ethod 2. $\left\{-\frac{3i}{2}, \frac{3i}{2}\right\}$

c. *M*ethod 4. $\left\{\frac{-3+\sqrt{17}}{4}, \frac{-3-\sqrt{17}}{4}\right\}$

d. Method 1. $\left\{\frac{1}{2}, -1\right\}$

Example 4:

a. $x^2-3x-10=0$

b. $10x^2+x-2=0$

c. $x^2+9=0$

Example 5: The age groups that can be expected to be involved in 3 fatal crashes per 100 million miles driven are ages 33 and 58.

Example 6:

a. 2.7 and –0.9

b. 0.1 and 2.1

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11.3 Quadratic Functions and Their Graphs

Graphs of Quadratic Functions

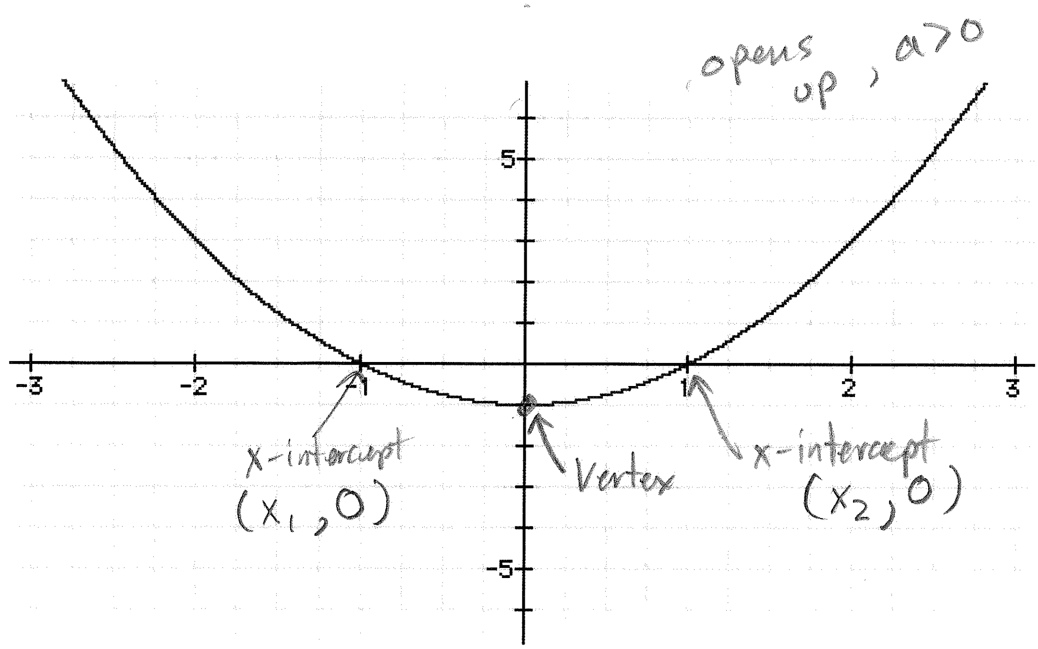
The graph of the quadratic function

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

is called a parabola.

Important features of parabolas are:

- The graph of a parabola is cup shaped.
- The graph opens upward if a > 0 and downward if a < 0.
- The vertex is the turning point of the parabola.
- If the parabola opens upward, the vertex is the lowest point on the graph.
- If the parabola opens downward, the vertex is the highest point on the graph.
- The graph of the parabola is symmetric to the vertical line that passes through its vertex.



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x-intercepts: $y=0$, $0 = (x-3)^2 - 1$
$1 + 0 = (x-3)^2 - 1 + 1$
$1 = (x-3)^2$
Either
$+\sqrt{1} = x-3$, or $-\sqrt{1} = x-3$
$1 = x-3$ $\quad -1 = x-3$

$3+1 = 3+x-3$, or $3+(-1) = 3+x-3$
$4 = x$ $\quad 2 = x$
$(4,0)$ & $(2,0)$

Graphing Quadratic Functions in the Form $f(x) = a(x-h)^2 + k$.

To graph $f(x) = a(x-h)^2 + k$:

1. Determine whether the parabola opens upward or downward. The graph opens upward if a > 0 and downward if a < 0.
2. Determine the vertex of the parabola. The vertex is (h,k).
3. Find any x-intercepts by replacing f(x) with 0. Solve the resulting quadratic equation for x. The x-intercepts are the points $(x_1,0)$ and $(x_2,0)$ where x_1 and x_2 are the solutions.
4. Find the y-intercept by replacing x with 0 and solving for y. The y-intercept is the point $(0, y_1)$ where y_1 is the solution.
5. Plot the intercepts and vertex and additional points as necessary. Connect these points with a smooth curve that is shaped like a cup.

Example 1: Graph $f(x) = (x-3)^2 - 1$

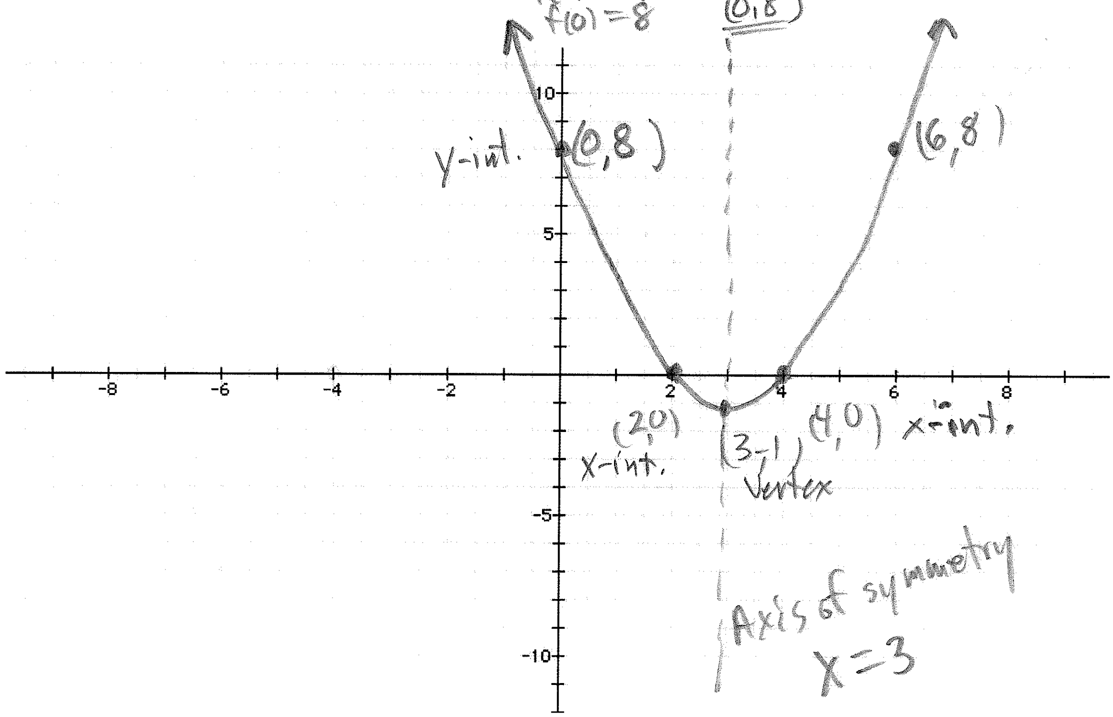
$f(x) = a(x-h)^2 + k = 1\cdot(x-3)^2 + (-1)$

$a > 0$ opens up $\longrightarrow a = 1$

vertex $\longrightarrow (h,k) = (3,-1)$

y-intercept $\longrightarrow x = 0$, $f(0) = [(0)-3]^2 - 1$
$f(0) = [-3]^2 - 1$
$f(0) = 9 - 1$
$f(0) = 8$

$(0,8)$



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x intercepts: $y=0$,

$0=(x-1)^2-4$

$4+0=(x-1)^2-4+4$

$4=(x-1)^2$

Either

$+\sqrt{4}=x-1$, or $-\sqrt{4}=x-1$

$2=x-1$, or $-2=x-1$

$2+1=x-1+1$ | $-2+1=x-1+1$

$3=x$ | $-1=x$

$(3,0)$ | $(-1,0)$

Example 2: Graph $f(x)=(x-1)^2-4$

$a=1$, $a>0$, opens up

Vertex: $(h,k)=(1,-4)$

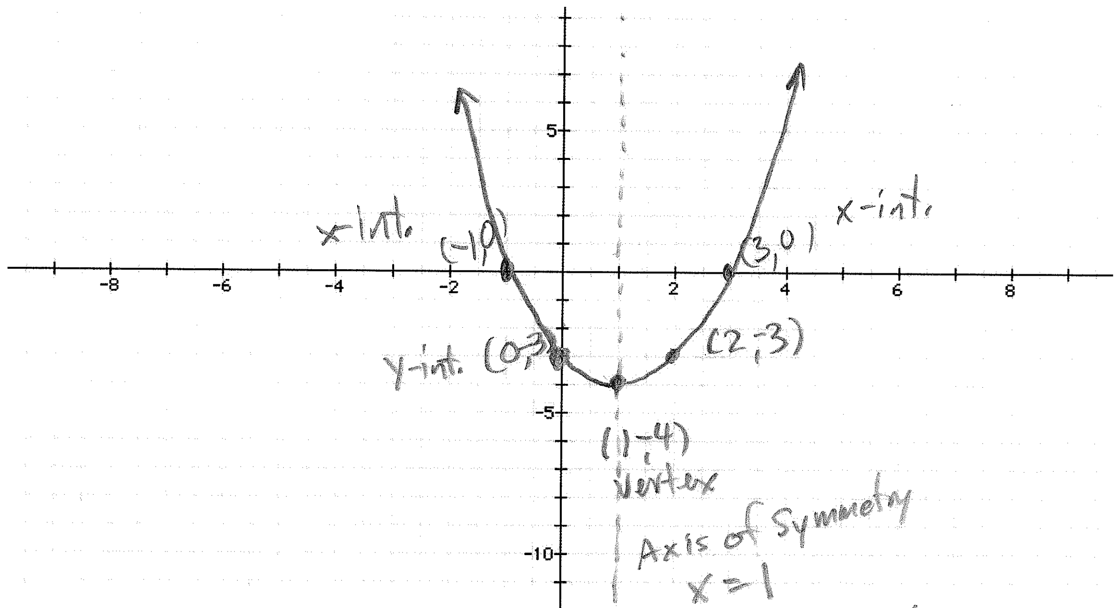
y-intercept: $(0,-3)$

$x=0$, $f(0)=[(0)-1]^2-4$

$f(0)=[-1]^2-4$

$f(0)=1-4$

$f(0)=-3$



Example 3: Graph $f(x)=-(x-1)^2+4$

$a=-1$, $a<0$, opens down

vertex: $(h,k)=(1,4)$

y-intercept: $(0,3)$

$x=0$, $f(0)=-[(0)-1]^2+4$

$f(0)=-[-1]^2+4$

$f(0)=-1+4$

$f(0)=3$

X-intercepts: $y=0$

$0=-[x-1]^2+4$

$(x-1)^2+0=-(x-1)^2+4+(x-1)^2$

$(x-1)^2=4$

Either

$x-1=+\sqrt{4}$, or $x-1=-\sqrt{4}$

$x-1=2$

$x-1+1=2+1$

$x=3$

OR

$x-1=-2$

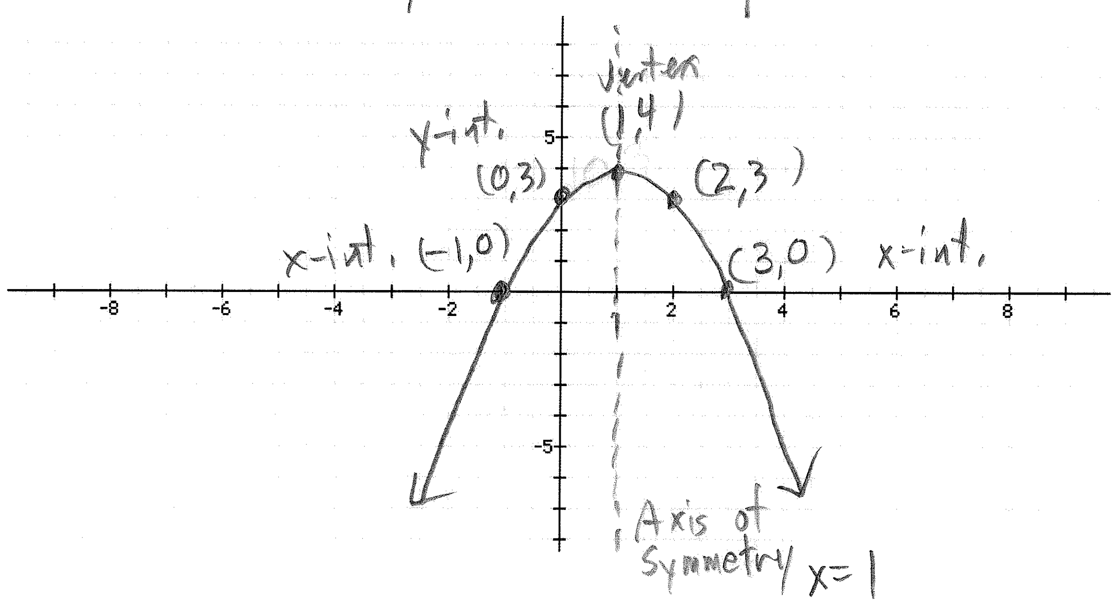
$x-1+1=-2+1$

$x=-1$

$(3,0)$

&

$(-1,0)$



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Example 4: Graph $f(x) = -2(x-3)^2 + 8$

$a=-2,\ a<0$, opens down

Vertex: $(3,8)$

y-intercept: $(0,-10)$

$x=0,\ f(0) = -2[(0)-3]^2+8$

$f(0) = -2[-3]^2+8$

$f(0) = -2\cdot 9+8$

$f(0) = -18+8$

$f(0) = -10$

x-intercept: $y=0$

$0 = -2(x-3)^2+8$

$2(x-3)^2+0 = -2(x-3)^2+8+2(x-3)^2$

$2(x-3)^2 = 8$

$\frac{1}{2}\cdot 2(x-3)^2 = \frac{1}{2}\cdot\frac{8}{1}$

$(x-3)^2 = 4$

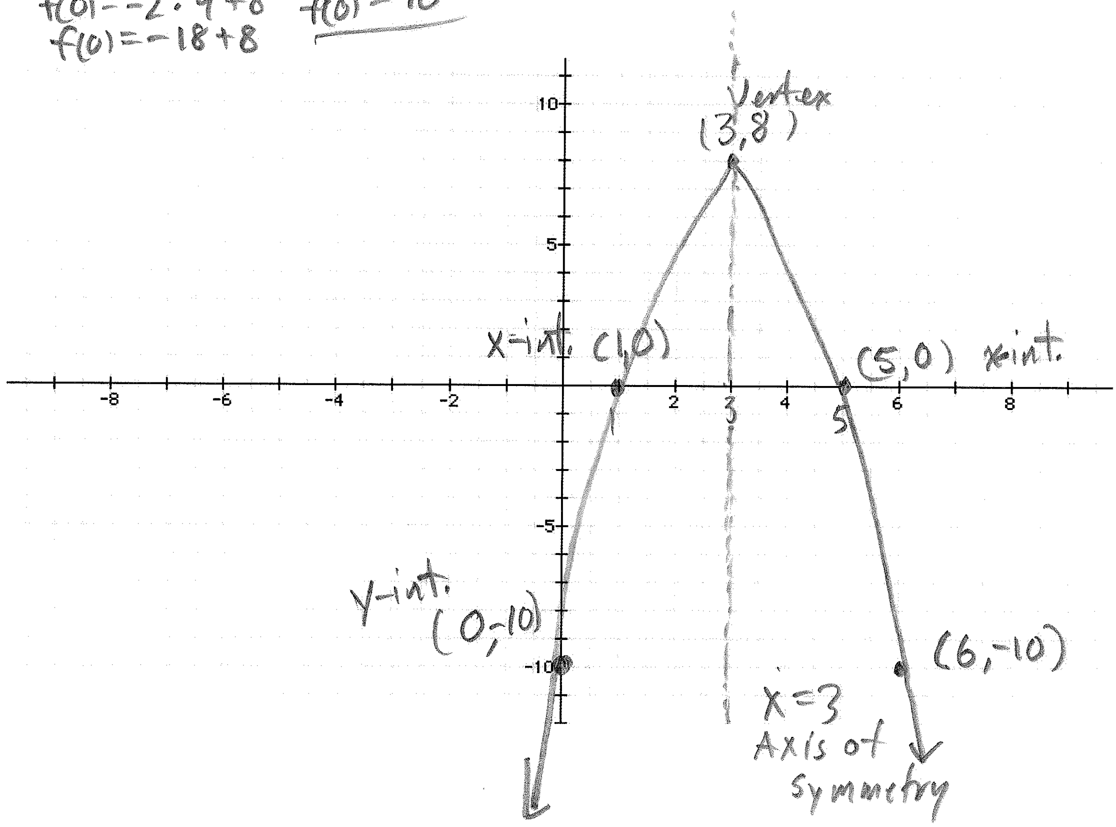
Either

$x-3 = +\sqrt{4}$, or $x-3 = -\sqrt{4}$

$x-3 = 2$, or $x-3=-2$

$x-3+3 = 2+3$	$x-3+3 = -2+3$
$x=5$	$x=1$

$(5,0)$ & $(1,0)$



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x-intercept: $y=0$

solve: $0=x^2-4x+3$

$0=(x-3)(x-1)$

Either

$x-3=0$, or $x-1=0$

$x-3+3=0+3$ | $x-1+1=0+1$

$x=3$ | $x=1$

$(3,0)$ & $(1,0)$

y-intercept: $x=0$

$y=f(0)$

$y=(0)^2-4(0)+3$

$y=0-0+3$

$y=3$

$(0,3)$

$\boxed{3 \quad 1,3}$

Graphing Quadratic Functions in the Form $f(x)=ax^2+bx+c$.

To graph $f(x)=ax^2+bx+c$:

1. Determine whether the parabola opens upward or downward. The graph opens upward if $a > 0$ and downward if $a < 0$.
2. Determine the vertex of the parabola. The vertex is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.
3. Find any x-intercepts by replacing f(x) with 0. Solve the resulting quadratic equation for x. The x-intercepts are the points $(x_1,0)$ and $(x_2,0)$ where x_1 and x_2 are the solutions.
4. Find the y-intercept by replacing x with 0 and solving for y. The y-intercept is the point $(0, y_1)$ where y_1 is the solution.
5. Plot the intercepts and vertex and additional points as necessary. Connect these points with a smooth curve that is shaped like a cup.

$\boxed{a=1 \quad b=-4 \quad c=3}$

Example 5: Graph $f(x)=x^2-4x+3$

$a=1$, $a>0$, opens up

vertex $= \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$x=\frac{-b}{2a}$

$x=\frac{-(-4)}{2(1)}$

$x=\frac{4}{2}$ & $y=f\left(\frac{-b}{2a}\right)$

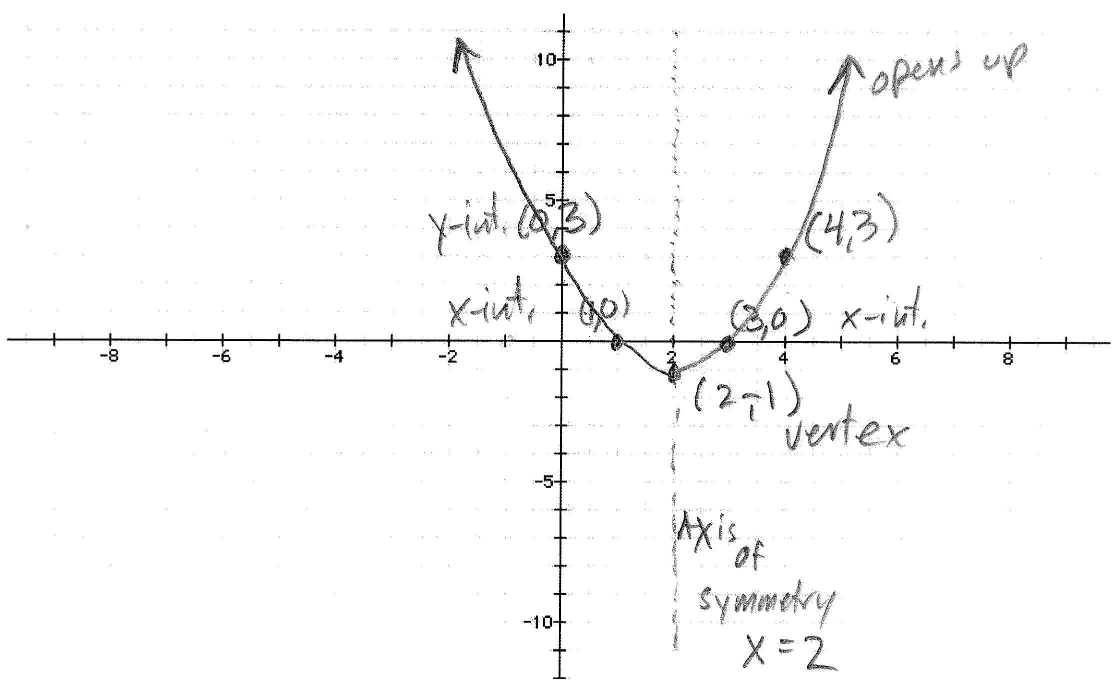
$x=2$ $\quad y=f(2)$

$y=(2)^2-4(2)+3$

$y=4-8+3$

$y=-1$

so, vertex $=(2,-1)$



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Example 6: Graph $f(x) = -x^2 - 2x + 3$

$\boxed{\begin{array}{l} a=-1 \\ b=-2 \\ c=3 \end{array}}$

$a=-1$, $a<0$, opens down

vertex $= \left(\frac{-b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

$x = \frac{-b}{2a}$

$x = \frac{-(-2)}{2(-1)}$

$x = -1$

$y = f\left(\frac{-b}{2a}\right)$

$y = f(-1)$

$y = -(-1)^2 - 2(-1) + 3$

$y = -1 + 2 + 3$

$y = 4$

So, vertex $= \underline{(-1, 4)}$

x-intercepts: $y=0$

Solve: $0 = -x^2 - 2x + 3$

$0 = -(x^2 + 2x - 3)$

$0 = -(x+3)(x-1)$

Either

$x+3=0$, or $x-1=0$

$-3+x+3=-3+0$ | $1+x-1=1+0$

$x=-3$ | $x=1$

$\underline{(-3,0)}$ & $\underline{(1,0)}$

y-intercept: $x=0$

$y = f(0)$

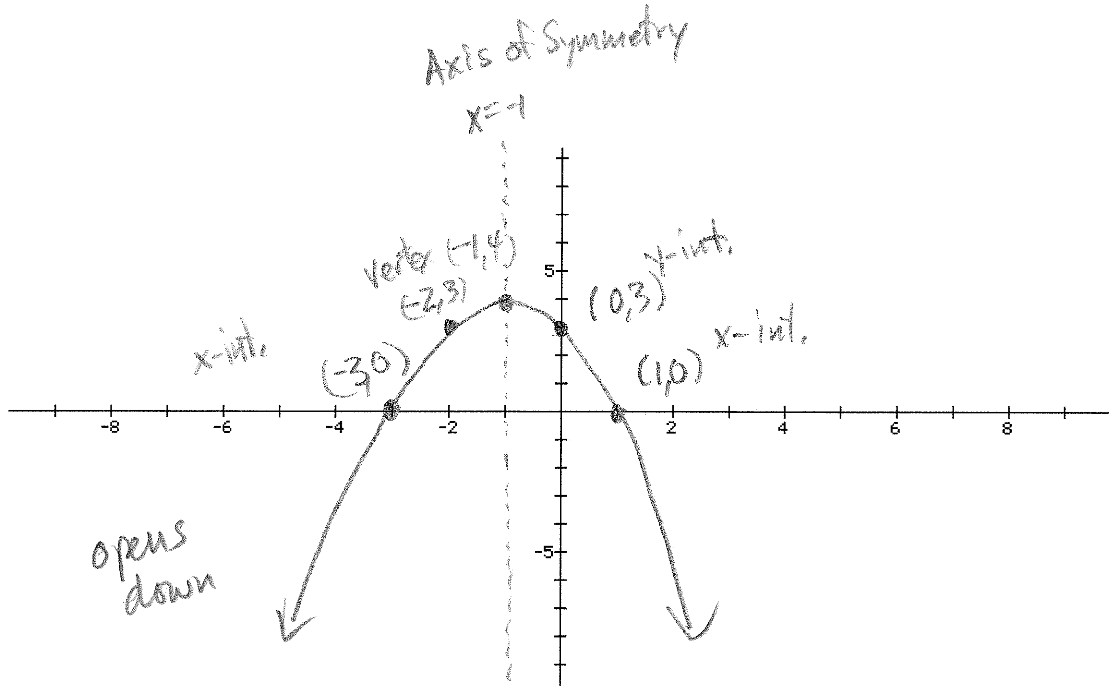
$y = -(0)^2 - 2(0) + 3$

$y = 0 - 0 + 3$

$y = 3$

$\underline{(0,3)}$

$\boxed{\begin{array}{c} 3 \\ -1, 3 \end{array}}$



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SPLUK

12 → 3, 4; 4 → 2, 2

$12 = 2^2 \cdot 3$

$12 = 4 \cdot 3$

24/46

a=-1
b=4
c=-1

Example 7: Graph $f(x) = -x^2 + 4x - 1$. Use your calculator to approximate the x-intercepts to the nearest tenth.

a=-1, a<0, opens down

Vertex $= \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

$x = \frac{-b}{2a}$

$x = \frac{-(4)}{2(-1)}$

$x = 2$

$y = f\left(\frac{-b}{2a}\right)$

$y = f(2)$

$y = -(2)^2 + 4(2) - 1$

$y = -4 + 8 - 1$

$y = 3$

So, vertex $= (2,3)$

x-intercepts: y=0

solve: $0 = -x^2 + 4x - 1$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(-1)(-1)}}{2(-1)}$

$x = \frac{-4 \pm \sqrt{16-4}}{-2}$

$x = \frac{-4 \pm \sqrt{12}}{-2}$

$x = \frac{-4 \pm \sqrt{4} \cdot \sqrt{3}}{-2}$

$x = \frac{-4 \pm 2\sqrt{3}}{-2}$

$x = \frac{-2(2 \mp \sqrt{3})}{-2 \cdot 1}$

$x = 2 \mp \sqrt{3}$

Either $x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$

$(2+\sqrt{3}, 0)$ & $(2-\sqrt{3}, 0)$

y-intercepts: x=0

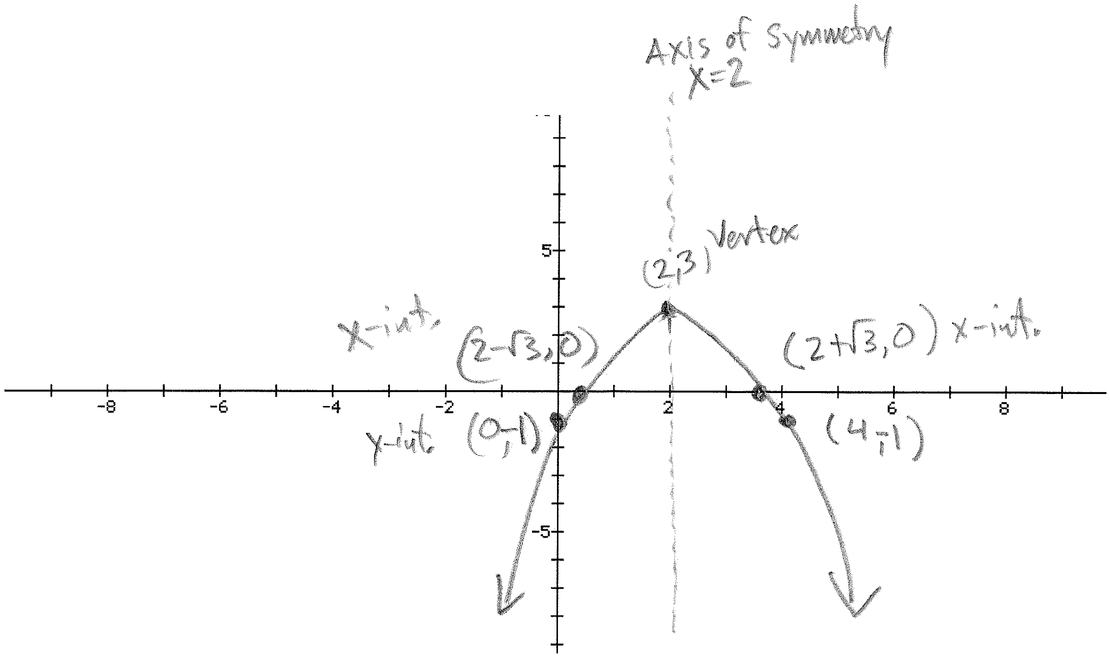
$y = f(0)$

$y = -(0)^2 + 4(0) - 1$

$y = -0 + 0 - 1$

$y = -1$

$(0, -1)$



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Applications of Quadratic Functions

Consider $f(x) = ax^2 + bx + c$.

1. If a > 0, then f has a minimum value that occurs at $x = -\frac{b}{2a}$.

   The minimum value is $f(-\frac{b}{2a})$.

2. If a < 0, then f has a maximum value that occurs at $x = -\frac{b}{2a}$.

   The maximum value is $f(-\frac{b}{2a})$.

Example 8: Use your calculator to find the maximum or minimum value for each of the following quadratic functions. Round to nearest tenth.

$\boxed{\begin{array}{l} a=1.2 \\ b=-4.1 \end{array}}$

a. $f(x) = 1.2x^2 - 4.1x + 2.2$

Minimum value $= f(\frac{-b}{2a})$

$x = \frac{-b}{2a}$

$x = \frac{-(-4.1)}{2(1.2)}$

$x = \frac{4.1}{2.4}$

$x \approx 1.7083$

$x \approx 1.7$

$y \approx f(1.7)$

$y \approx 1.2(1.7)^2 - 4.1(1.7) + 2.2$

$y \approx 1.2(2.89) - 6.97 + 2.2$

$y \approx 3.468 - 4.77$

$y \approx -1.302$

$y \approx -1.3$

The minimum value is approximately -1.3.

$\boxed{\begin{array}{l} a=-1.3 \\ b=6.1 \end{array}}$

b. $f(x) = -1.3x^2 + 6.1x - 6$

Minimum value $= f(\frac{-b}{2a})$

$x = \frac{-b}{2a}$

$x = \frac{-(6.1)}{2(-1.3)}$

$x = \frac{6.1}{2.6}$

$x \approx 2.346$

$x \approx 2.3$

$y \approx f(2.3)$

$y \approx -1.3(2.3)^2 + 6.1(2.3) - 6$

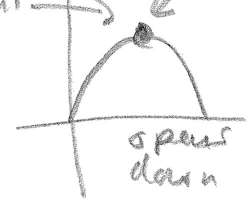
$y \approx -1.3(5.29) + 14.03 - 6$

$y \approx -6.877 + 8.03$

$y \approx 1.153$

$y \approx 1.2$

The minimum value is approximately 1.2.



$a=-16 \quad b=64$, $a>0$, opens down

Example 9: A person standing on the ground throws a ball into the air. The quadratic function

$$s(t) = -16t^2 + 64t$$

models the ball's height above the ground, s(t), in feet, t seconds after it has been thrown. What is the maximum height that the ball reaches?

The maximum height of the ball occurs at the vertex of the parabola. Find $S\left(-\frac{b}{2a}\right)$ to find the maximum height.

$t = \frac{-b}{2a}$

$t = \frac{-(64)}{2(-16)}$

$t = \frac{64}{32}$

$t = 2$

$y = S\left(\frac{-b}{2a}\right)$

$y = S(2)$

$y = -16(2)^2 + 64(2)$

$y = -16 \cdot 4 + 128$

$y = -64 + 128$

$\underline{y = 64}$

The maximum height the ball reaches after 2 seconds is 64 feet.

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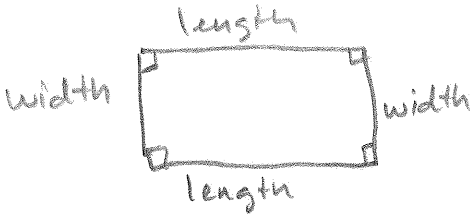


$P = 100$ yd, $P = 2l + 2w$

$100 = 2x + 2w$

$-2x + 100 = -2x + 2x + 2w$

$-2x + 100 = 2w$

$\frac{1}{2} \cdot (-2x + 100) = \frac{1}{2} \cdot \frac{2w}{1}$

$\boxed{w = -x + 50}$

In some verbal problems, the quadratic functions are not given, but must be formed. In these cases, follow the strategy below to solve the problem.

Strategy For Solving Problems Involving Maximizing or Minimizing Quadratic Functions

1. Read the problem carefully and decide which quantity is to be maximized or minimized.
2. Use the conditions of the problem to express the quantity as a function in one variable.
3. Rewrite the function in the form $f(x) = ax^2 + bx + c$.
4. If a > 0, f has a minimum value at $x = -\frac{b}{2a}$. If a < 0, f has a maximum value at $x = -\frac{b}{2a}$.
5. Answer the question posed in the problem.

Example 10: You have 100 yards of fencing to enclose a rectangular region. Find the dimensions of the rectangle that maximize the enclosed area. What is the maximum area?

let x = length of rectangular region

Area = $l \cdot w$

$A(x) = (x)(-x + 50)$

$A(x) = -x^2 + 50x$

$\boxed{\begin{array}{l} a = -1 \\ b = 50 \end{array}}$

Find the maximum value:

$x = \frac{-b}{2a}$

$x = \frac{-(50)}{2(-1)}$

$x = \frac{-50}{-2}$

$x = 25$ yd

length Dimension

$A\left(\frac{-b}{2a}\right) = A(25)$

$A(25) = -(25)^2 + 50(25)$

$A(25) = -625 + 1,250$

$A(25) = 625$ yd^2

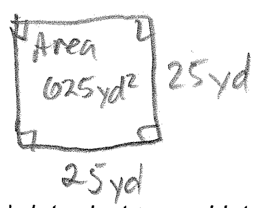
Maximum Area

ANS: With 100 yds of fencing, you can enclose a rectangular area of 625 yd^2 by making a square with dimensions of 25 yds by 25 yds.

width Dimension

width $= -x + 50$

$= -(25) + 50 = 25$ yd



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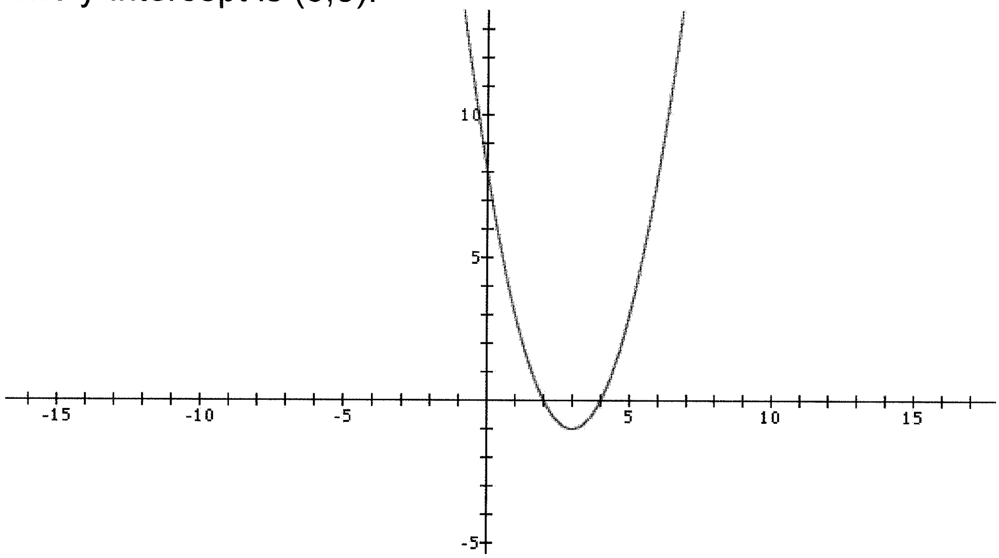
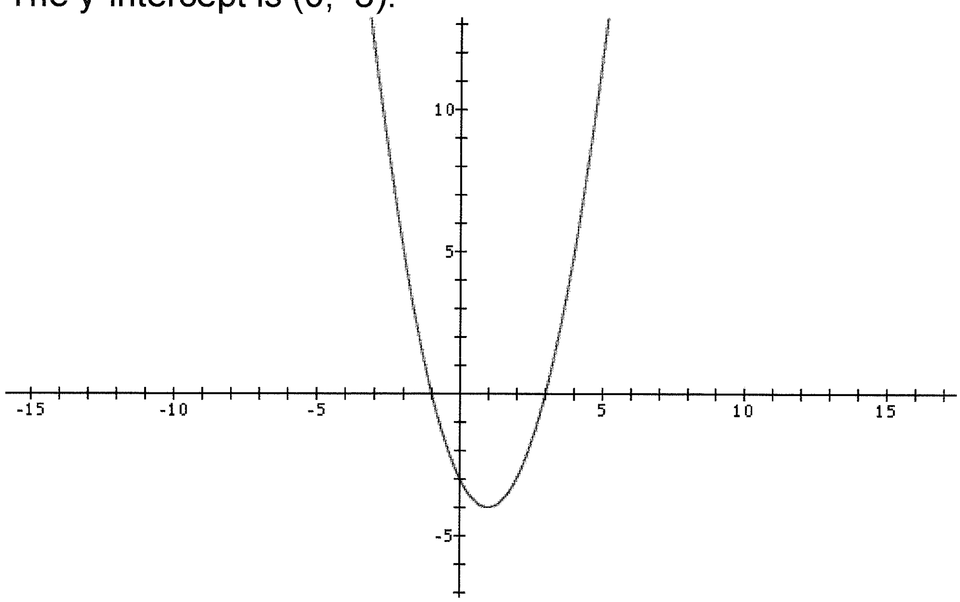
Answers Section 11.3

Example 1:
The parabola opens upward.
The vertex is (3, –1)
The x-intercepts are (4,0) and (2,0).
The y-intercept is (0,8).



Example 2:
The parabola opens upward.
The vertex is (1, –4)
The x-intercepts are (3,0) and (–1,0).
The y-intercept is (0,–3).



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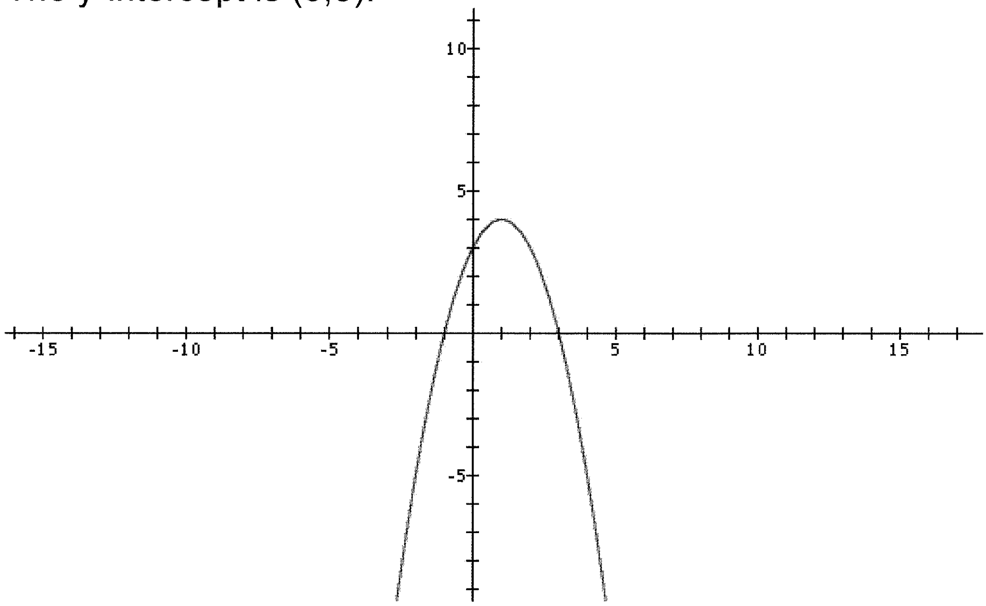
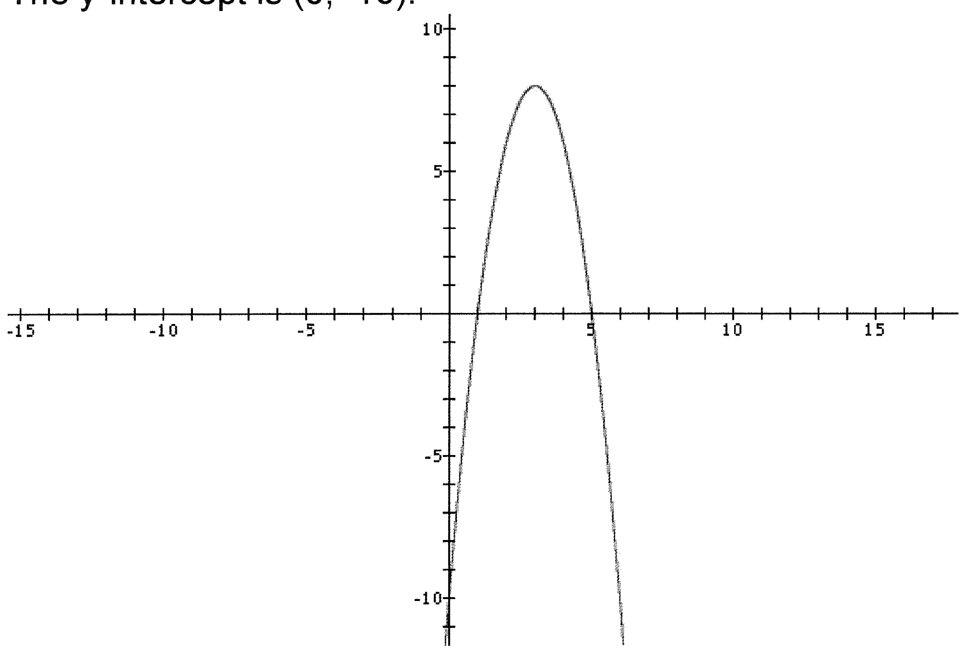
Example 3:

The parabola opens downward.
The vertex is (1, 4)
The x-intercepts are (3,0) and (–1,0).
The y-intercept is (0,3).



Example 4:

The parabola opens downward.
The vertex is (3, 8)
The x-intercepts are (5,0) and (1,0).
The y-intercept is (0,–10).



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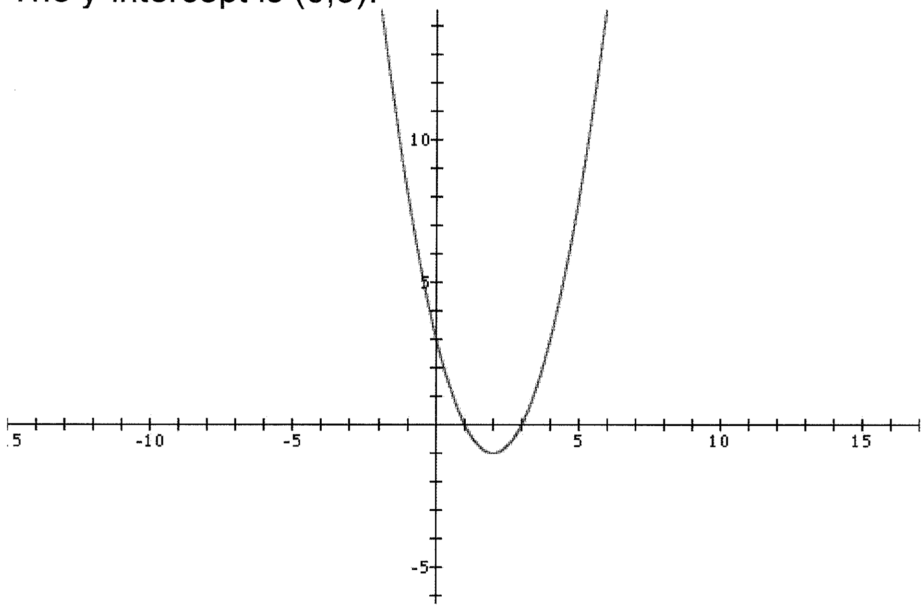
Example 5:

The parabola opens upward.

The vertex is (2, –1)

The x-intercepts are (1,0) and (3,0).

The y-intercept is (0,3).



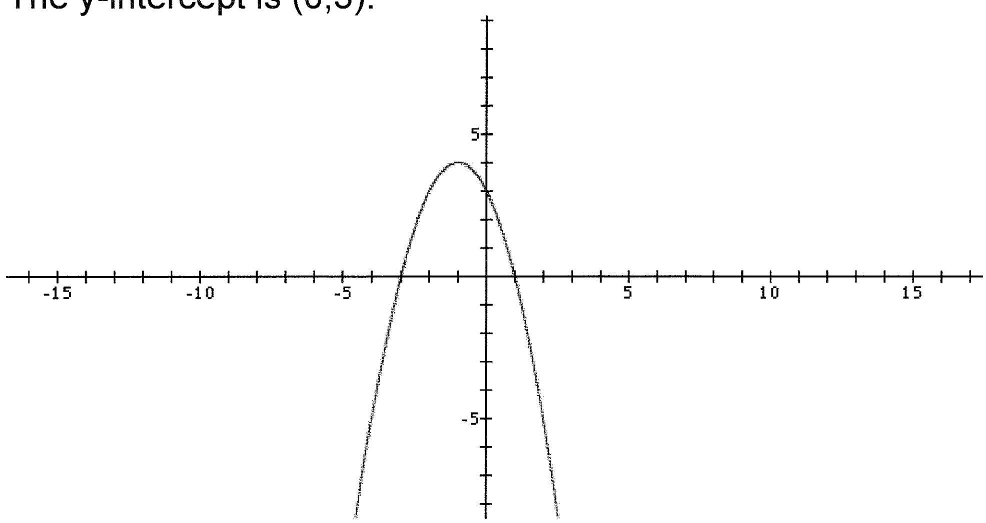
Example 6:

The parabola opens downward.

The vertex is (–1,4)

The x-intercepts are (–3,0) and (1,0).

The y-intercept is (0,3).



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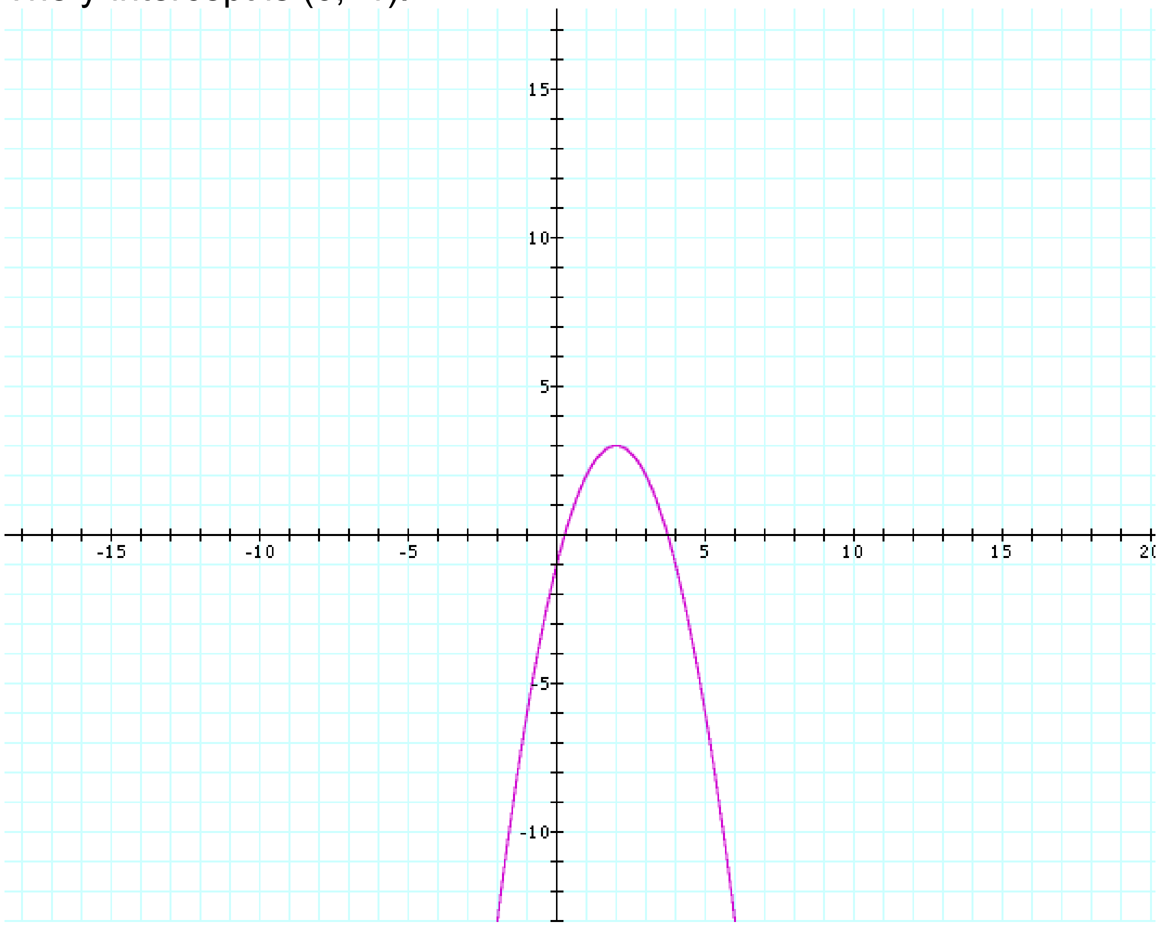
Example 7:
The parabola opens downward.
The vertex is (2, 3)
The x-intercepts are $\left(2 \pm \sqrt{3}, 0\right)$ or approx. (3.7,0) and (0,0.3)

The y-intercept is (0,–1).



Example 8:
a. Minimum value is –1.3.
b. Maximum value is 1.2.

Example 9: The maximum height is 64 feet (the y-coordinate of the vertex).

Example 10: The dimensions of the rectangle of maximum area are 25 yards by 25 yards. The maximum area is 625 square yards.

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11.4 Equations in Quadratic Form

Quadratic Form

An equation that is quadratic in form is an equation that can be expressed as a quadratic equation using an appropriate substitution. In symbols:

- equation in quadratic form $ax^{2n}+bx^{n}+c=0$
- substitution $t=x^{n}$
- resulting quadratic equation: $at^{2}+bt+c=0$

Example 1: Choose an appropriate substitution and write the given equations as a quadratic equation in t.

a. $x^4-10x^2+9=0$; $t=x^2$, $t^2=(x^2)^2=x^4$
$t^2-10t+9=0$

b. $x^{\frac{1}{2}}-10x^{\frac{1}{4}}+9=0$; $t=x^{\frac{1}{4}}$, $t^2=(x^{\frac{1}{4}})^2=x^{\frac{2}{4}}=x^{\frac{1}{2}}$
$t^2-10t+9=0$

c. $2x-\sqrt{x}-10=0$; $t=\sqrt{x}=x^{\frac{1}{2}}$, $t^2=(\sqrt{x})^2=(x^{1/2})^2=x$
$2t^2-t-10=0$

d. $(x+3)^2+7(x+3)-18=0$; $t=x+3$, $t^2=(x+3)^2$
$t^2+7t-18=0$

e. $x^{-2}-x^{-1}-6=0$; $t=x^{-1}$, $t^2=(x^{-1})^2=x^{-2}$
$t^2-t-6=0$

Solving Equations That Are Quadratic in Form

To solve equations that are quadratic in form:
1. Choose an appropriate substitution and rewrite the original equation as a quadratic equation in t.
2. Solve the quadratic equation in t.
3. Use the original substitution and the t-solutions to find the x-solutions.
4. Check your solutions. If at any time during the solution process you raised both sides of an equation to an even power, a check is required, since raising both sides to an even power may introduce extraneous solutions.

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Example 2: Solve the given equations.

a. $x^4 - 10x^2 + 9 = 0$

let $t = x^2$, $t^2 = (x^2)^2 = x^4$

$t^2 - 10t + 9 = 0$

$(t-9)(t-1) = 0$

9
1, 9
3, 3

Either

$t - 9 = 0$, or $t - 1 = 0$

$9 + t - 9 = 9 + 0$ | $1 + t - 1 = 1 + 0$

$t = 9$ | $t = 1$

$x^2 = 9$ | $x^2 = 1$

Either

$x = +\sqrt{9}$, or $x = -\sqrt{9}$

$x = 3$ | $x = -3$

Either

$x = +\sqrt{1}$, or $x = -\sqrt{1}$

$x = 1$ | $x = -1$

$\boxed{\text{The solution set is } \{1, -1, 3, -3\}.}$

check:

$x = 1$

$(1)^4 - 10(1)^2 + 9 = 0$

$1 - 10 \cdot 1 + 9 = 0$

$1 - 10 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

$x = -1$

$(-1)^4 - 10(-1)^2 + 9 = 0$

$1 - 10 \cdot (1) + 9 = 0$

$1 - 10 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

$x = 3$

$(3)^4 - 10(3)^2 + 9 = 0$

$81 - 10 \cdot (9) + 9 = 0$

$81 - 90 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

$x = -3$

$(-3)^4 - 10(-3)^2 + 9 = 0$

$81 - 10 \cdot (9) + 9 = 0$

$81 - 90 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

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Let $t = x^3$, $t^2 = (x^3)^2 = x^6$

b. $x^6 - 10x^3 + 9 = 0$

9
1, 9
3, 3

$t^2 - 10t + 9 = 0$

$(t - 9)(t - 1) = 0$

Either

$t - 9 = 0$, or $t - 1 = 0$

$9 + t - 9 = 9 + 0$ | $1 + t - 1 = 1 + 0$

$t = 9$ | $t = 1$

$x^3 = 9$ | $x^3 = 1$

$x = \sqrt[3]{9}$ | $x = \sqrt[3]{1}$

$x = 1$

$\boxed{\text{The solution set is } \{1, \sqrt[3]{9}\}.}$

check!

$x = 1$

$(1)^6 - 10(1)^3 + 9 = 0$

$1 - 10 \cdot (1) + 9 = 0$

$1 - 10 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

$x = \sqrt[3]{9}$

$(\sqrt[3]{9})^6 - 10(\sqrt[3]{9})^3 + 9 = 0$

$(9^{1/3})^6 - 10(9^{1/3})^3 + 9 = 0$

$9^{6/3} - 10(9^{3/3}) + 9 = 0$

$9^2 - 10 \cdot 9 + 9 = 0$

$81 - 90 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

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SDWK

$9^4 = 9 \cdot 9 \cdot 9 \cdot 9$

$= 81 \cdot 81$

$= 6{,}561$

let $t = x^{\frac{1}{4}}$, $t^2 = (x^{\frac{1}{4}})^2 = x^{\frac{2}{4}} = x^{\frac{1}{2}}$

c. $x^{\frac{1}{2}} - 10x^{\frac{1}{4}} + 9 = 0$

9

1, 9

3, 3

$t^2 - 10t + 9 = 0$

$(t - 9)(t - 1) = 0$

Either

$t - 9 = 0$, or $t - 1 = 0$

$9 + t - 9 = 9 + 0$

$t = 9$

$x^{\frac{1}{4}} = 9$

$(x^{\frac{1}{4}})^4 = (9)^4$

$x = 6{,}561$

$1 + t - 1 = 1 + 0$

$t = 1$

$x^{\frac{1}{4}} = 1$

$(x^{\frac{1}{4}})^4 = (1)^4$

$x = 1$

The solution set is $\{6{,}561, 1\}$.

Check:

$x = 6{,}561$

$(6{,}561)^{\frac{1}{2}} - 10(6{,}561)^{\frac{1}{4}} + 9 = 0$

$\sqrt{(81)^2} - 10 \cdot \sqrt[4]{9^4} + 9 = 0$

$81 - 10 \cdot 9 + 9 = 0$

$81 - 90 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

$x = 1$

$(1)^{1/2} - 10(1)^{1/4} + 9 = 0$

$\sqrt{1} - 10 \cdot \sqrt[4]{1} + 9 = 0$

$1 - 10 \cdot 1 + 9 = 0$

$1 - 10 + 9 = 0$

$-9 + 9 = 0$

$0 = 0$

TRUE!

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let $t=\sqrt{x}$, $t^2=(\sqrt{x})^2=x$

d. $2x-\sqrt{x}-10=0$

$2t^2-t-10=0$

10
1, 10
2, 5

$(2t-5)(t+2)=0$

Either

$2t-5=0$, or $t+2=0$

$5+2t-5=5+0$

$2t=5$

$\frac{2t}{2}=\frac{5}{2}$

$t=\frac{5}{2}$

$\sqrt{x}=\frac{5}{2}$

$(\sqrt{x})^2=\left(\frac{5}{2}\right)^2$

$x=\frac{25}{4}$

$-2+t+2=-2+0$

$t=-2$

Look!

$\sqrt{x}=-2$

Look!

$(\sqrt{x})^2=(-2)^2$

$x=4$

check

$x=4$

$2(4)-\sqrt{4}-10=0$

$8-2-10=0$

$6-10=0$

$-4=0$

False!

$x=\frac{25}{4}$

$\frac{2}{1}\left(\frac{25}{4}\right)-\sqrt{\frac{25}{4}}-10=0$

$\frac{25}{2}-\frac{5}{2}-10=0$

$\frac{20}{2}-10=0$

$10-10=0$

$0=0$

TRUE!

$\boxed{\text{The solution set is } \left\{\frac{25}{4}\right\}.}$

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let $t = (x+3)$, $t^2 = (x+3)^2$

e. $(x+3)^2 + 7(x+3) - 18 = 0$

18
1, 18
2, 9
3, 6

$t^2 + 7t - 18 = 0$

$(t+9)(t-2) = 0$

Either

$t+9=0$, or $t-2=0$

$-9+t+9=-9+0$

$t=9$

$x+3=-9$

$-3+x+3=-3+(-9)$

$x=-12$

$2+t-2=2+0$

$t=2$

$x+3=2$

$-3+x+3=-3+2$

$x=-1$

The solution set is $\{-12, -1\}$.

check

$x=-1$

$[(-1)+3]^2 + 7[(-1)+3] - 18 = 0$

$[2]^2 + 7[2] - 18 = 0$

$4 + 14 - 18 = 0$

$18 - 18 = 0$

$0 = 0$

TRUE!

$x=-12$

$[(-12)+3]^2 + 7[(-12)+3] - 18 = 0$

$[-9]^2 + 7[-9] - 18 = 0$

$81 - 63 - 18 = 0$

$18 - 18 = 0$

$0 = 0$

TRUE!

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Let $t = x^{-1}$, $t^2 = (x^{-1})^2 = x^{-2}$

$f.\ x^{-2} - x^{-1} - 6 = 0$

$t^2 - t - 6 = 0$

6
1, 6
2, 3

$(t+2)(t-3) = 0$

Either

$t+2=0$, or $t-3=0$

$-2+t+2=-2+0$

$\underline{t=-2}$

$x^{-1}=-2$

$(x^{-1})^{-1}=(-2)^{-1}$

$x^1 = \frac{1}{-2}$

$x = -\frac{1}{2}$

$3+t-3=3+0$

$\underline{t=3}$

$x^{-1}=3$

$(x^{-1})^{-1}=3^{-1}$

$x^1 = \frac{1}{3}$

$x = \frac{1}{3}$

The solution set is $\{-\frac{1}{2}, \frac{1}{3}\}$.

check!

$x = -\frac{1}{2}$

$(-\frac{1}{2})^{-2} - (-\frac{1}{2})^{-1} - 6 = 0$

$(-\frac{2}{1})^2 - (-\frac{2}{1})^1 - 6 = 0$

$(-2)^2 - (-2) - 6 = 0$

$4 + 2 - 6 = 0$

$6 - 6 = 0$

$0 = 0$

TRUE!

$x = \frac{1}{3}$

$(\frac{1}{3})^{-2} - (\frac{1}{3})^{-1} - 6 = 0$

$(\frac{3}{1})^2 - (\frac{3}{1})^1 - 6 = 0$

$(3)^2 - (3) - 6 = 0$

$9 - 3 - 6 = 0$

$6 - 6 = 0$

$0 = 0$

TRUE!

Finding x-intercepts of a Quadratic-in-Form Function

To find x-intercepts of a function, substitute 0 for f(x) and solve the resulting equation.

Example 3: Find the x-intercepts of the given functions.

a. $f(x) = x^4 - 13x^2 + 36$

Find x-intercepts: $y = 0$

Solve: $0 = x^4 - 13x^2 + 36$

let $t = x^2$, $t^2 = (x^2)^2 = x^4$

$0 = t^2 - 13t + 36$

$0 = (t-9)(t-4)$

either

$t - 9 = 0$, or $t - 4 = 0$

36
1, 36
2, 18
3, 12
4, 9
6, 6

$9 + t - 9 = 9 + 0$ | $4 + t - 4 = 4 + 0$

$t = 9$ | $t = 4$

$x^2 = 9$ | $x^2 = 4$

either | either

$x = +\sqrt{9}$, or $x = -\sqrt{9}$ | $x = +\sqrt{4}$, or $x = -\sqrt{x}$

$x = 3$ $x = -3$ | $x = 2$ $x = -2$

$(3,0)$ & $(-3,0)$ & $(2,0)$ & $(-2,0)$

The x-intercepts are the four points: $(3,0)$, $(-3,0)$, $(2,0)$, and $(-2,0)$.

check:

$f(3) = (3)^4 - 13(3)^2 + 36$
$f(3) = 81 - 13 \cdot 9 + 36$
$f(3) = 81 - 117 + 36$
$f(3) = 0$ ✓

$f(-3) = (-3)^4 - 13(-3)^2 + 36$
$f(-3) = 81 - 13 \cdot 9 + 36$
$f(-3) = 81 - 117 + 36$
$f(-3) = 0$

$f(2) = (2)^4 - 13(2)^2 + 36$
$f(2) = 16 - 13 \cdot 4 + 36$
$f(2) = 16 - 52 + 36$
$f(2) = 0$

$f(-2) = (-2)^4 - 13(-2)^2 + 36$
$f(-2) = 16 - 13 \cdot 4 + 36$
$f(-2) = 16 - 52 + 36$
$f(-2) = 0$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

b. $f(x) = x^{\frac{2}{3}} - 9x^{\frac{1}{3}} + 8$

Find x-intercepts: $y = 0$

Solve: $0 = x^{2/3} - 9x^{1/3} + 8$

Let $t = x^{1/3}$, $t^2 = (x^{1/3})^2 = x^{2/3}$

$0 = t^2 - 9 + 8$

$0 = (t - 8)(t - 1)$

$\boxed{\begin{matrix} 8 \\ 1, 8 \\ 2\ 4 \end{matrix}}$

Either

$t - 8 = 0$, or $t - 1 = 0$

$8 + t - 8 = 8 + 0$	$1 + t - 1 = 1 + 0$
$t = 8$	$t = 1$
$x^{1/3} = 8$	$x^{1/3} = 1$
$(x^{1/3})^3 = (8)^3$	$(x^{1/3})^3 = (1)^3$
$x^{3/3} = 512$	$x = 1$
$x = 512$	
$(512, 0)$	$(1, 0)$

check

$f(512) = (512)^{2/3} - 9(512)^{1/3} + 8$

$f(512) = (\sqrt[3]{512})^2 - 9 \cdot \sqrt[3]{512} + 8$

$f(512) = (\sqrt[3]{8^3})^2 - 9 \cdot \sqrt[3]{8^3} + 8$

$f(512) = (8)^2 - 9 \cdot 8 + 8$

$f(512) = 64 - 72 + 8$

$f(512) = 0$ ✓

$f(1) = (1)^{2/3} - 9(1)^{1/3} + 8$

$f(1) = (\sqrt[3]{1})^2 - 9 \sqrt[3]{1} + 8$

$f(1) = (1)^2 - 9 \cdot 1 + 8$

$f(1) = 1 - 9 + 8$

$f(1) = 0$ ✓

The x-intercepts are the two points (512,0) and (1,0).

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

Answers Section 11.4

Example 1:

a. Let $t=x^2$. $t^2-10t+9=0$

b. Let $t=x^{\frac{1}{4}}$. $t^2-10t+9=0$

c. Let $t=\sqrt{x}$. $2t^2-t-10=0$

d. Let $t=(x+3)$. $t^2+7t-18=0$

e. Let $t=x^{-1}$. $t^2-t-6=0$

Example 2:

a. $\{-3,-1,1,3\}$

b. $\{1,\sqrt[3]{9}\}$

c. $\{1,6561\}$

d. $\left\{\frac{25}{4}\right\}$

e. $\{-1,-12\}$

f. $\left\{-\frac{1}{2},\frac{1}{3}\right\}$

Example 3:

a. x-intercepts are $(\pm 2,0)$ and $(\pm 3,0)$

b. x-intercepts are (1,0) and (512,0)

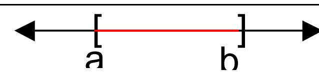
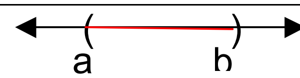
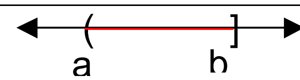
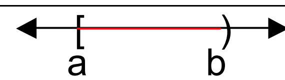
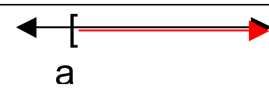
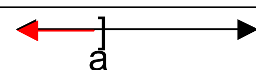
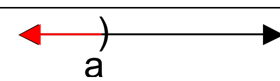
Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer

11.5 Polynomial and Rational Inequalities

Interval Notation-Review

Intervals can be expressed in interval notation, set-builder notation or graphically on the number line. The following chart shows the different notations. You may use interval notation, inequality notation or set-builder notation to depict intervals.

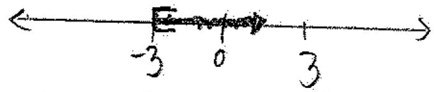
Let a and b represent two real numbers with a < b.

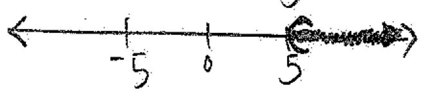
Type of Interval	Interval Notation	Set-Builder Notation	Graph on the Number Line
Closed Interval	$[a,b]$	$\{x \mid a \le x \le b\}$	
Open Interval	(a,b)	$\{x \mid a < x < b\}$	
Half-Open Interval	$(a,b]$	$\{x \mid a < x \le b\}$	
Half-Open Interval	$[a,b)$	$\{x \mid a \le x < b\}$	
Interval That Is Not Bounded on the Right	$[a,\infty)$	$\{x \mid a \le x < \infty\}$ or $\{x \mid x \ge a\}$	
Interval That Is Not Bounded on the Right	(a,∞)	$\{x \mid a < x < \infty\}$ or $\{x \mid x > a\}$	
Interval That Is Not Bounded on the Right	$(-\infty,a]$	$\{x \mid -\infty < x \le a\}$ or $\{x \mid x \le a\}$	
Interval That Is Not Bounded on the Right	$(-\infty,a)$	$\{x \mid -\infty < x < a\}$ or $\{x / x < a\}$	
Interval That Is Not Bounded on the Right	$(-\infty,\infty)$	$\{x \mid -\infty < x < \infty\}$ or {x|x is a real no.}	

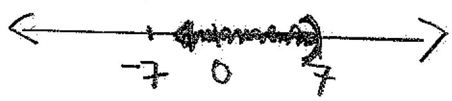
Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

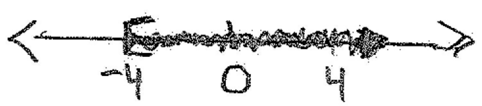
*** Review From chapter 2, section $\boxed{2.7}$

Example 1: Write each inequality in interval notation.

a. $x \geq -3$  ; $[-3, \infty)$

b. $5 < x < \infty$  ; $(5, \infty)$

c. $x < 7$  ; $(-\infty, 7)$

d. $-4 \leq x < \infty$  ; $[-4, \infty)$

Example 2: Write each interval in set-builder notation.

a. $[-4,\infty)$ $= \{x \mid -4 \leq x < \infty\}$ or $\{x \mid x \geq -4\}$

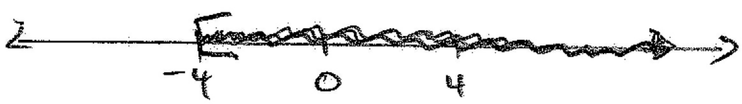
b. $(-\infty,5)$ $= \{z \mid -\infty < z < 5\}$ or $\{z \mid z < 5\}$

c. $(-7, -2]$ $= \{k \mid -7 < k \leq -2\}$

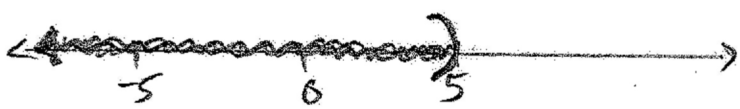
d. $(-1,4)$ $= \{j \mid -1 < j < 4\}$

Example 3: Graph each interval on the number line.

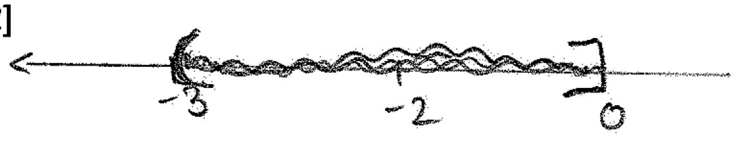
a. $[-4,\infty)$



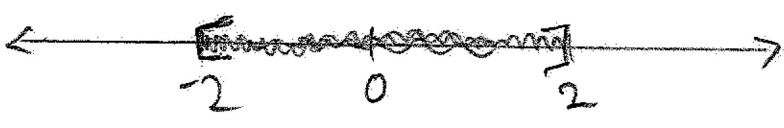
b. $(-\infty,5)$



c. $(-3, -2]$



e. $[-2,2]$



Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Polynomial Inequalities

Definition of a Polynomial Inequality

A polynomial inequality is any inequality that can be put in one of the forms

$$f(x) > 0 \qquad f(x) \geq 0$$
$$f(x) < 0 \qquad f(x) \leq 0$$

where f(x) is a polynomial. Recall that a polynomial is a single term or the sum or difference of terms all of which have variables in numerators only and which have only whole number exponents.

Test $x = 4$
$(4)^2 - 2(4) - 3 \geq 0$
$16 - 8 - 3 \geq 0$
$8 - 3 \geq 0$
$5 \geq 0$
TRUE!

Solving Polynomial Inequalities

Solutions to a polynomial inequality

- $f(x) > 0$ consists of the x-values for which the graph of f(x) lies above the x-axis.
- $f(x) \geq 0$ consists of the x-values for which the graph of f(x) lies above the x-axis or is touching or crossing the x-axis.
- $f(x) < 0$ consists of the x-values for which the graph lies below the x-axis.
- $f(x) \leq 0$ consists of the x-values for which the graph lies below the x-axis or is touching or crossing the x-axis.

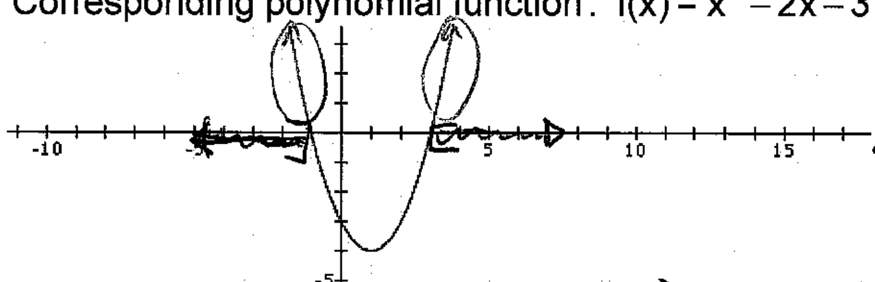
Thus the x-values at which the graph moves from below-to-above or above-to-below the x-axis are crucial values. These x-values are the solutions to the equation $f(x) = 0$. They are **boundary points** for the inequality.

Example 4: Solve the given inequality by using the graph of the corresponding polynomial function. $y = f(x)$

Inequality: "$x^2 - 2x - 3 \geq 0$" "above $y = 0$"

Corresponding polynomial function: $f(x) = x^2 - 2x - 3$

$x^2 - 2x - 3 = 0$
$(x+1)(x-3) = 0$
Either
$x + 1 = 0$ or $x - 3 = 0$
$x = -1$ | $x = 3$
Boundary! Boundary!



Solution: ? $(-\infty, -1] \cup [3, \infty)$

$-1 \quad 0 \quad 3$

Test $x = -2$
$(-2)^2 - 2(-2) - 3 \geq 0$
$4 + 4 - 3 \geq 0$
$8 - 3 \geq 0$
$5 \geq 0$
TRUE!

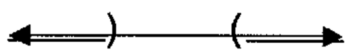
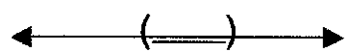
Test $x = 0$
$(0)^2 - 2(0) - 3 \geq 0$
$-3 \geq 0$
False!

Test $x = 4$
look above ↑

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Procedure for Solving Polynomial Inequalities Algebraically

1. Express the inequality in the standard form $f(x) > 0$ or $f(x) < 0$.
2. Solve the equation $f(x) = 0$. The real solutions are the boundary points.
3. Locate these boundary points on a number line, thereby dividing the number line into test intervals. If the inequality symbol is "<" or ">", exclude all boundary points from the test intervals.
4. Choose one representative number within each test interval. If substituting that value into the original inequality produces a true statement, then all real numbers in the test interval belong to the solution set. If substituting that value into the original inequality produces a false statement, then no real number in the test interval belongs to the solution set.
5. Write the solution set, selecting the interval(s) that produced a true statement. The graph of the solution set on a number line usually appears as

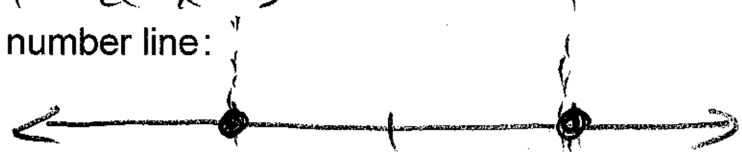
 or 

Example 5: Solve the given inequality.

a. $x^2 - 2x - 3 \ge 0$

Boundary points: $x = -1$ & $x = 3$

Graph boundary points on a number line:



Identify intervals and complete chart:

Intervals	Representative Number	Substitute into Inequality	Conclusion
$(-\infty, -1)$	-2	$(-2)^2 - 2(-2) - 3 \ge 0$ $5 \ge 0$	True. Thus $(-\infty, -1]$ belongs to sol'n set
$(-1, 3)$	0	$(0)^2 - 2(0) - 3 \ge 0$ $-3 \ge 0$	False!
$(3, \infty)$	4	$(4)^2 - 2(4) - 3 \ge 0$ $16 - 8 - 3 \ge 0$ $5 \ge 0$	TRUE! $[3, \infty)$

Write the solution in interval notation.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

b. $(x+1)(x-2)(x+4) < 0$

$(x+1)(x-2)(x+4) = 0$

$\boxed{1^{st}}$ Find Boundary points using "="

Either

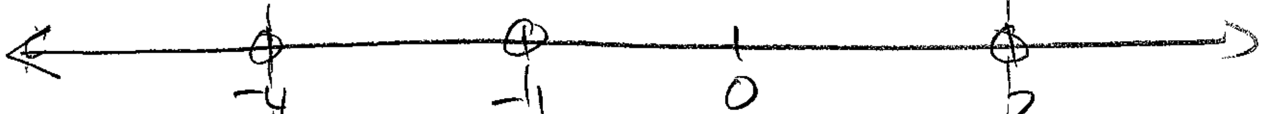
$x+1=0$, $x-2=0$, or $x+4=0$

$-1+x+1=-1+0$ $\quad$ $2+x-2=2+0$ $\quad$ $-4+x+4=-4+0$

$\underline{x=-1}$ $\quad$ $\underline{x=2}$ $\quad$ $\underline{x=-4}$

$\boxed{2nd}$ Use Boundary Points to "partition" the number line, inorder to test the possible solution regions

$(-\infty,-4)$ Region $\quad$ $(-4,-1)$ Region $\quad$ $(-1,2)$ Region $\quad$ $(2,\infty)$ Region



Test $\underline{x=-5}$

$(-5+1)(-5-2)(-5+4)<0$

$(-4)(-7)(-1)<0$

$-28<0$

TRUE!

Test $\underline{x=-2}$

$(-2+1)(-2-2)(-2+4)<0$

$(-1)(-4)(2)<0$

$8<0$

False!

Test $x=0$

$(0+1)(0-2)(0+4)<0$

$(1)(-2)(4)<0$

$-8<0$

TRUE!

Test $x=3$

$(3+1)(3-2)(3+4)<0$

$(4)(1)(7)<0$

$28<0$

False!

$(-\infty,-4)$ is part of the solution set

$(-4,-1)$ is not part of the solution set.

$(-1,2)$ is part of the solution set

$(2,\infty)$ is not part of the solution set.

$\boxed{(-\infty,-4)\cup(-1,2) \text{ is the solution set.}}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Solving Rational Inequalities

A rational inequality is an inequality that can be put in one of the forms:

$$\frac{P(x)}{Q(x)} \le 0 \quad \frac{P(x)}{Q(x)} \ge 0$$

$$\frac{P(x)}{Q(x)} < 0 \quad \frac{P(x)}{Q(x)} > 0$$

Procedure for Solving Rational Inequalities:

1. Write the inequality so that one side is zero and the other side is a single quotient.
2. Find the boundary points by setting the numerator and the denominator equal to zero.
3. Locate the boundary points on a number line.
4. Use the boundary points to establish test intervals. If the inequality symbol is “<” or “>”, exclude all boundary points from the test intervals. Also, exclude any boundary points that make the denominator equal to zero.

5. Take one representative number within each test interval and substitute that number into the original inequality to determine if the inequality is true or false at that representative number.

5. The solution set consists of the intervals that produced a true statement.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

Example 6: Solve the given inequality. Write your answers in interval notation.

a. $\frac{x+5}{x+2} < 0$

Find the boundary points!

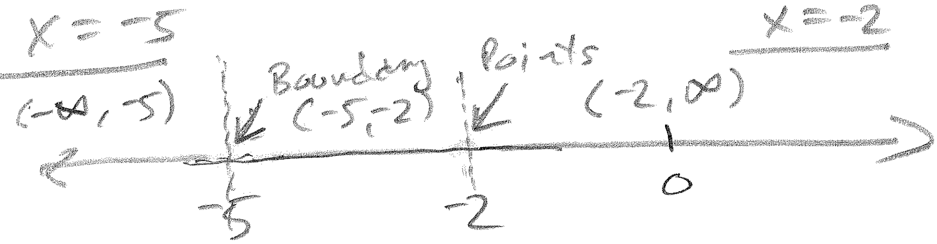
I solve! $x+5=0$ AND II Solve: $x+2=0$

$-5+x+5=-5+0$

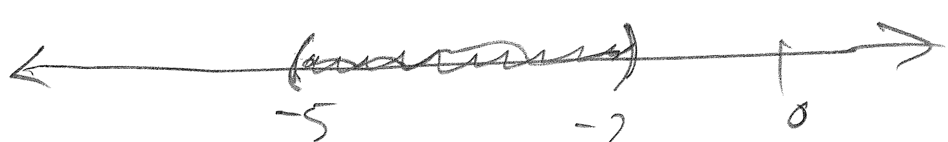
$x=-5$

$-2+x+2=-2+0$

$x=-2$



Test Intervals:	$(\infty,-5)$	$(-5,-2)$	$(-2,\infty)$
Representative value	$x=-6$	$x=-3$	$x=0$
Test	$\frac{(-6)+5}{(-6)+2}<0$ $\frac{-1}{-4}<0$ $\frac{1}{4}<0$ False!	$\frac{(-3)+5}{(-3)+2}<0$ $\frac{2}{-1}<0$ $-2<0$ TRUE!	$\frac{(0)+5}{(0)+2}<0$ $\frac{5}{2}<0$ False!



$\boxed{\text{The solution set is } (-5,-2).}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

b. $\dfrac{x}{x+2} \geq 2$

$$\frac{-2}{1} + \frac{x}{x+2} \geq -2 + 2$$

$$\frac{-2}{1} \cdot \left(\frac{x+2}{x+2}\right) + \frac{x}{x+2} \geq 0$$

$$\frac{-2x-4}{x+2} + \frac{x}{x+2} \geq 0$$

$$\frac{-2x-4+x}{x+2} \geq 0$$

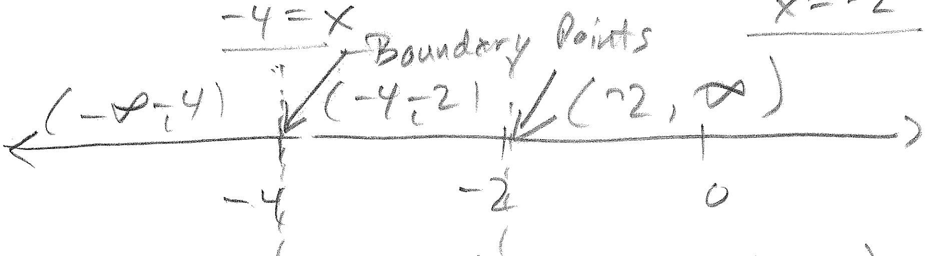
$$\frac{-x-4}{x+2} \geq 0$$ ✓

Find the boundary points:

(I) solve: $-x-4=0$ AND (II) Solve: $x+2=0$

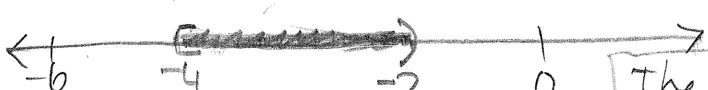
$-x-4+x=0+x$ $\qquad$ $-2+x+2=-2+0$

$-4=x$ $\qquad$ $x=-2$



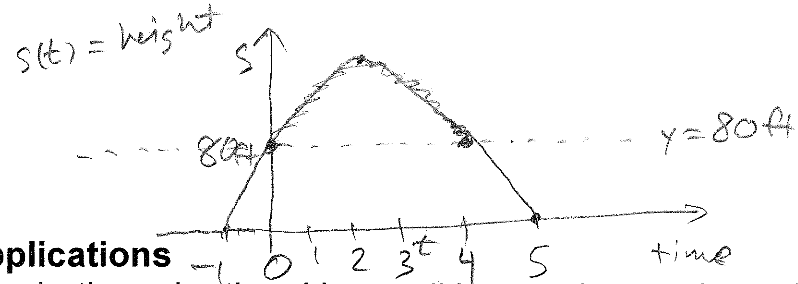
Test Intervals: $(-\infty,-4)$ $(-4,-2)$ $(-2,\infty)$

Representative Value	$x=-5$	$x=-3$	$x=0$
Test	$\frac{(-5)}{(-5)+2} \geq 2$ $\frac{-5}{-3} > 2$ $\frac{5}{3} > 2$ False!	$\frac{(-3)}{(-3)+2} \geq 2$ $\frac{-3}{-1} \geq 2$ $3 \geq 2$ TRUE!	$\frac{(0)}{(0)+2} \geq 2$ $\frac{0}{2} \geq 2$ $0 \geq 2$ False!



$\boxed{\text{The solution set is } [-4,-2).}$

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.



Applications

Quadratic and rational inequalities can be used to solve applied problems.

Example 7: A model rocket is launched from the top of a cliff 80 feet above sea level. The function

$$s(t) = -16t^2 + 64t + 80$$

models the rocket's height above the water, s(t), in feet, t seconds after it was launched. During which time period will the rocket's height exceed that of the cliff?

Height of the cliff = 80ft

$s(t) > 80\text{ft}$ ← Solve: "height exceeds that of the cliff"

Solve for t: $-16t^2 + 64t + 80 > 80$

$-80 + (-16t^2 + 64t + 80) > -80 + 80$

$-16t^2 + 64t > 0$ ✓

Find the boundary points:

(I) Solve: $-16t^2 + 64t = 0$, AND (II) there is no equation from the denominator,

$-16(t^2 - 4t) = 0$

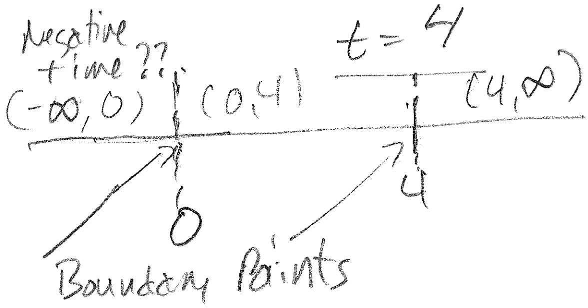
$-16(t)(t-4) = 0$

Either

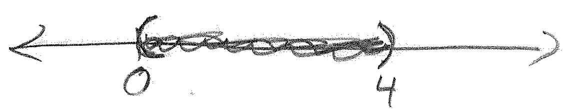
$t = 0$, or $t - 4 = 0$

$4 + t - 4 = 4 + 0$

$t = 4$



Test Intervals:	$(0, 4)$	$(4, \infty)$
Representative Value	$t = 2$	$t = 6$
Test	$-16(2)^2 + 64(2) > 0$ $-16(4) + 128 > 0$ $-64 + 128 > 0$ $64 > 0$ TRUE!	$-16(6)^2 + 64(6) > 0$ $-16(36) + 384 > 0$ $-576 + 384 > 0$ $-192 > 0$ False!



The solution set is $(0, 4)$.

ANS: The rocket is above the 80ft cliff between 0 and 4 seconds.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.

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Answers Section 11.5

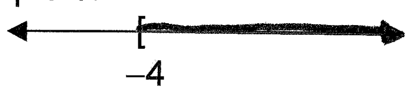
Example1:

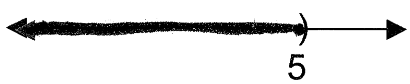
a. $[-3, \infty)$
b. $(5, \infty)$
c. $(-\infty, 7)$
d. $[-4, \infty)$

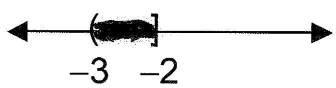
Example 2:

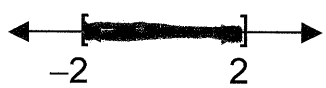
a. $\{x|x \geq -4\}$ or $\{x| -4 \leq x < \infty\}$
b. $\{x|x < 5\}$ or $\{x| -\infty \leq x < 5\}$
c. $\{x| -7 < x \leq -2\}$
d. $\{x| -1 < x < 4\}$

Example 3:

a. 

b. 

c. 

d. 

Example 4: $(-\infty, -1] \cup [3, \infty)$

Example 5:

a. $(-\infty, -1] \cup [3, \infty)$
b. $(-\infty, -4) \cup (-1, 2)$

Example 6:

a. $(-5, -2)$
b. $[-4, -2)$

Example 7: (0,4) The rocket is above the cliff between 0 and 4 seconds.

Note: Portions of this document are excerpted from the textbook *Introductory and Intermediate Algebra for College Students* by Robert Blitzer.